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Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

299

Proceedings of the Second Conference on Compact Transformation Groups

University of Massachusetts, Amherst, 1971

Part II

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COMMENTS BY THE EDITORS

The Second Conference on Compact Transformation Groups was held at the University of Massachusetts, Amherst from June 7 to June 18, 1971 under the sponsorship of the Advanced Science Education Program of the National Science Foundation. There were a total of 70 participants at the conference.

As was the case at the first conference at Tulane University in 1967, the emphasis continued to be on differentiable transformation groups. In this connection there was a continued application of surgery typified by the lectures of Browder, Shaneson, and Yang (joint work with Montgomery). A new feature was the applications of the Atiyah-Singer Index Theorem to differentiable transformation groups typified by the lectures of Hinrichsen, Petrie, and Rothenberg. In connection with topological and algebraic methods significant innovations were made by Raymond (joint work with Conner) in the construction of manifolds admitting no effective finite group action, by R. Lee in studying free actions of finite groups on spheres using ideas and methods derived from algebraic K-theory and by Su (joint work with W.Y. Hsiang) in applying the notion of geometric weight systems developed recently by W. Y. Hsiang. There were several lectures on algebraic varieties by Michael Davis, Peter Orlik, and Philip Wagreich. Interest in this area arose from the application several years ago of Brieskorn varieties to the study of actions on homotopy spheres.

These Proceedings contain not only material presented at the conference but also articles received by the editors up to the summer of 1972. We have divided the articles into two volumes; the first volume is devoted to smooth techniques while the second to non-smooth techniques. While the proper assignment of a few papers was not obvious, the editors felt that this classification offered, in general, the most natural division of the material.

H. T. Ku
L. N. Mann
J. L. Sicks
J. C. Su

Amherst, Mass., July 1972

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W. Browder	Equivariant Differential Topology
M. Davis	Actions on Exotic Stiefel Manifolds
D. Erle	On Unitary and Symplectic Knot Manifolds
I. Fary	Group Action and Betti Sheaf
L. Feldman	Reducing Bundles in Differentiable G-spaces
V. Giambalvo	Cobordism of Line Bundles with Restricted Characteristic Class
R. Goldstein	Free Differentiable Circle Actions on 6-Manifolds
M. Goto	Orbits of One-Parameter Groups
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J. Hinrichsen	Operators Elliptic Relative to Group Actions
S. Illman	Equivariant Singular Homology
L. Kauffman	Cyclic Branched Covering Spaces and O(n)-Actions
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H. T. Ku	Characteristic Invariants of Free Differentiable Actions of S^1 and S^3 on Homotopy Spheres
K. W. Kwun	Transfer Homomorphisms of Whitehead Groups of Some Cyclic Groups
R. Lee	Semi-Characteristic Classes
	The Topological Period of Periodic Groups
S. López de Medrano	Cobordism of Diffeomorphisms of (k-1)-Connected 2k-Manifolds
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INTRODUCTORY REMARKS

The subject of transformation groups is in an active period and it is good for all of us interested to meet and exchange ideas at first hand. A generation ago the fewer people then working in a field could manage to keep in touch by correspondence or occasional contacts at general meetings, but this is now more difficult, and specialized conferences of this kind perform an important service not easily achieved in any other way. Transformation groups is an area of topology which has connections with most of the other areas of topology. In the past, progress in any part of topology has often led to progress in transformation groups. This is likely to continue and all of us must keep as well informed as we can about what others are doing at the same time as we are continuing with our own problems. Conversely transformation groups has sometimes contributed to other areas, at the very least by suggesting questions and problems. It is a great pleasure to attend a conference on a very interesting subject under such convenient conditions and congenial surroundings as have been provided here.

Deane Montgomery

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MANIFOLDS WITH FEW PERIODIC HOMEOMORPHISMS

by P. E. Conner* and Frank Raymond*

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1. INTRODUCTION

In this paper we construct, in §8, a family of distinct compact connected 4-manifolds $V(k)$, $k > 1$, with the property that every finite group must act trivially on $V(k)$. The boundary of each of the $V(k)$ is the 3-sphere and hence the distinct open 4-manifolds $U(k) = V(k) - \partial(V(k))$ also possesses a total lack of non-trivial periodic homeomorphisms.

The result is obtained by adjoining 4-dimensional cells along the boundaries of each $V(k)$ which yield distinct closed aspherical manifolds, $B(k)$, (that is, closed manifolds which are $K(\pi,1)$'s). What we show is that each non-trivial periodic homeomorphism of $B(k)$ has no fixed points. Now, since any self homeomorphism h of $V(k)$ may be extended to $B(k)$ by introducing a fixed point at the center of the added 4-cell, h cannot have finite period unless it is the identity.

Let $\{n_1,\ldots,n_\ell\}$ be any set of distinct positive integers. Let D_{n_i} be the dihedral group given by $0 \to Z_{n_i} \to D_{n_i} \overset{\leftarrow}{\to} Z_2 \to 0$. In Section 7 we construct a closed aspherical $(2\ell+1)$-manifold $M^{2\ell+1}(n_1,\ldots,n_\ell)$ for each $\{n_1,n_2,\ldots,n_\ell\}$. Let (G,M) denote an effective action of a finite group G on M. We show that G must be a subgroup of $D_{n_1} \oplus \cdots \oplus D_{n_\ell}$. In particular, (7.2), $M^3(1)$ is a non-orientable closed aspherical 3-manifold for which G must be $Z_2 \cong D_1$ and must have exactly 2 circles of fixed points. This is the closest we have been able to get to the trivial group for a closed

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*Supported in part by the National Science Foundation

manifold.

Let (G,X) be an action of a group G on a path connected space X and $x \in X$ a base point. If x is a fixed point then there is a homomorphism $\theta : G \to \mathrm{Aut}\, \pi_1(X,x)$ given by $\theta(g) = g_* : \pi_1(X,x) \to \pi_1(X,x)$. Let the outer automorphisms of $\pi_1(X,x)$, the automorphisms of $\pi_1(X,x)$ modulo the inner automorphisms, be denoted by $\mathrm{Out}(\pi_1(X,x))$. Even if x is not left fixed by G, there still exists a homomorphism $\Psi : G \to \mathrm{Out}(\pi_1(X,x))$, §3.

Basic for the estimation of the size of finite effective G in the examples cited when X is a closed connected aspherical manifold are the following:

1.1. *If* $x \in F(G,X)$, *the fixed point set, then* θ *is a monomorphism.*

1.2. *If* $\pi_1(X,x)$ *has trivial center then* Ψ *is a monomorphism.*

A proof of 1.1. appears in [5; 6.2], and also here in A.11. The second theorem is an unpublished result of A. Borel. A proof of Borel's theorem, suitable for our purpose, will be given in §3. In the Appendix and §7 we extend 1.1. and 1.2. as part of the Smith theory for actions of p-groups on aspherical manifolds.

The claim of freeness for $(G,B(k))$ is achieved by showing that $\mathrm{Aut}(\pi_1(B(k),x))$ has no elements of finite order other than the identity, §8.

The results on $M^{2\ell+1}(n_1,\ldots,n_\ell)$ come from the fact that

$$\mathrm{Out}(\pi_1(M)) \cong Z^{\ell-1} \oplus D_{n_1} \oplus \ldots \oplus D_{n_\ell},$$

and that the center of $\pi_1(M)$ is trivial, §6 and §7.

Let us explain how such a calculation is made. Take a group π and a homomorphism $\Phi : Z \to \mathrm{Aut}\, \pi$ and form the semi-direct product $L = \pi \circ_\Phi Z$. In Section 4 we develop a method for calculating $\mathrm{Aut}(L)$ and $\mathrm{Out}(L)$ in terms of $\mathrm{Aut}(\pi)$, $\mathrm{Out}(\pi)$, knowledge of the cyclic group

generated by Φ in $\mathrm{Aut}\,\pi$ and $\mathrm{Out}\,\pi$, and the center K of π. Under suitable assumptions we find L has trivial center and the sequence

$$0 \to H_0^{\Phi}(Z;K) \to \mathrm{Out}(L) \to N(\Phi)/(\Phi) \to 1$$

is exact where N denotes the normalizer of the group, (Φ), generated by Φ in $\mathrm{Out}(\pi)$. The sequence is split if π is abelian. Section 5 tells us that the contribution of the non-zero elements of $H_0^{\Phi}(Z;K)$ can only arise from automorphisms of infinite order. One now tries to find a closed aspherical manifold Y^n whose fundamental group is π and a homeomorphism h, with fixed point $y \in Y^n$, so that $\Phi = h_* : \pi_1(Y^n,y) \to \pi_1(Y^n,y)$. One may then construct a closed aspherical manifold M^{n+1} as a fiber bundle over the circle S^1, fiber Y^n and structure group the cyclic group generated by Φ in $\mathrm{Out}(\pi_1(Y^n))$ where $\pi_1(M^{n+1}) = L = \pi_1(Y) \circ_{\Phi} Z$.

In Section 6 we look at $\pi = Z^k$ and consider certain $\Phi \in GL(k,Z)$. Fundamental for our calculations is the matrix $\Phi_n = \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix} \in GL(2,Z)$, $n \geq 1$. We show that

$$0 \to Z_n \to \mathrm{Out}(Z \oplus Z \circ_{\Phi_n} Z) \overset{\leftarrow}{\to} Z_2 \to 0,$$

is split exact and that the action of Z_2 on Z_n is multiplication by -1. Hence, $\mathrm{Out}(L(n)) = \mathrm{Out}(Z \oplus Z \circ_{\Phi_n} Z) = D_n$. Note that $\Phi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathrm{Out}(L(1)) = Z_2$. The specific manifolds $M^{2\ell+1}(n_1,\ldots,n_\ell)$, and others similar to them, are produced from Y equal to the 2ℓ-dimensional torus with $\Phi \in GL(2\ell,Z)$ coming from blocks of 2×2 matrices Φ_n along the diagonal. The 4-manifolds $B(k)$ arise by taking for Y^3 principal circle bundles over the 2-torus with Euler class $2k$, and a suitable Φ.

Our interest here has been in the action of finite groups G. If, on the other hand, G is assumed to be a compact, connected Lie group then there are a number of results known that guarantee that (G,M)

must be trivial. For example, if M is closed and aspherical, G must be a torus T^k, [5], the Euler characteristic $\chi(M^n) = 0$, [4], and the rank of the center of $\pi_1(M^n)$ must be greater than or equal to k. Obviously these criteria yield many examples where G must be trivial to be effective. Other examples are any connected sum of closed oriented 3-manifolds where one of the factors has fundamental group not cyclic. Also, Atiyah and Hirzebruch have shown [1] that orientable closed 4k-dimensional spin manifolds which admit non-trivial smooth circle actions have $\hat{A}$ genus 0. We do not use, however, any smoothness assumptions throughout this paper.

We would like to express our appreciation to Professor Borel for having shown us his result, (1.2.). Its use is fundamental for our work.

2. REALIZATION OF GROUP EXTENSIONS

We shall be concerned with a group extension

$$1 \to N \to G \to F \to 1.$$

We shall write N additively. We recall that G is the set $N \times F$ with the group operation given by

$$(g_1,x) \cdot (g_2,y) = (g_1 + \varphi(x)(g_2) + f(x,y), xy)$$

wherein

(i) $\varphi : F \to \mathrm{Aut}(N)$ is a function with $\varphi(e) =$ identity

(ii) $f : F \times F \to N$ is a function satisfying

(a) $\varphi(x)(\varphi(y)(g)) = f(x,y) + (\varphi(xy)(g)) - f(x,y)$

(b) $f(x,e) = f(e,x) = 0$

(c) $\varphi(z)(f(x,y)) + f(z,xy) = f(z,x) + f(zx,y)$.

Our primary intention is the application of the following.

2.1. Lemma: If $h : N \to L$ is a homomorphism then h can be ex-

tended to a homomorphism $H : G \to L$ *if and only if there is a function* $T : F \to L$ *satisfying*

$$T(x) \cdot h(g) = h(\varphi(x)(g)) \cdot T(x)$$
$$T(x) \cdot T(y) = h(f(x,y)) \cdot T(xy).$$

Proof: Suppose first that such an extension exists, $H : G \to L$. Put $T(x) = H(0,x) \in L$. Then $T(x) \cdot T(y) = H((0,x) \cdot (0,y)) =$ $= H(f(x,y),xy) = H(f(x,y),e) \cdot H((0,xy)) = h(f(x,y)) \cdot T(xy)$. In addition $T(x) \cdot h(g) = H((0,x) \cdot (g,e)) = H((\varphi(x)(g),x) = H(\varphi(x)(g),e)H((0,x))$ $= h(\varphi(x)(g)) \cdot T(x)$.

Conversely, if such a function T exists then we put

$$H(g,x) = h(g) \cdot T(x).$$

Then $H(g_1 + \varphi(x)(g_2) + f(x,y),xy) = h(g_1) \cdot h(\varphi(x)(g_2)) \cdot h(f(x,y)) \cdot T(xy) = h(g_1) \cdot h(\varphi(x)(g_2)) \cdot T(x) \cdot T(y) = h(g_1) \cdot T(x) \cdot h(g_2) \cdot T(y) = H(g_1,x) \cdot H(g_2,y)$.

We shall apply this lemma when L is a group of homeomorphisms on a space.

Suppose now (F,X) is a group of homeomorphisms on a pathwise connected, locally pathwise connected space which is also semi-locally-1-connected. Select a base point $a \in X$ and proceed to define a group extension

$$1 \to \pi_1(X,a) \to G \to F \to 1$$

as follows. For each $x \in F$ choose a path $P_x(t)$ in X issuing from a with $P_x(1) = x \cdot a$. We assume $P_e(t) \equiv a$.

First define $\varphi : F \to \mathrm{Aut}(\pi_1(X,a))$. If $\sigma(t)$ is a loop based at a, then $\varphi(x)(\sigma)$ is represented by

$$\begin{cases} P_x(3t) & 0 \le t \le \frac{1}{3} \\ x\sigma(3t-1) & \frac{1}{3} \le t \le \frac{2}{3} \\ P_x(3-3t) & \frac{2}{3} \le t \le 1 \end{cases}$$

For any pair (x,y) we denote by $f(x,y) \varepsilon \pi_1(X,a)$ the element represented by

$$\begin{cases} P_x(3t) \\ x \cdot P_y(3t-1) \\ P_{xy}(3-3t). \end{cases}$$

It is clear that $f(x,e) = f(e,x) = 0$. Let us consider $f(z,x) + f(zx,y)$.

This is represented by the sum of paths

$$\begin{cases} P_z(3t) \\ zP_x(3t-1) \\ P_{zx}(3-3t) \end{cases} + \begin{cases} P_{zx}(3t) \\ zxP_y(3t-1) \\ P_{zxy}(3-3t). \end{cases}$$

This must be compared with $\varphi(z)(f(x,y)) + f(z,xy)$, which is represented by

$$\begin{cases} P_z(3t) \\ zP_x(3(3t-1)) \\ zxP_y(9t-4) \\ zP_{xy}(3(2-3t)) \\ P_z(3-3t) \end{cases} + \begin{cases} P_z(3t) \\ zP_{xy}(3t-1) \\ P_{zxy}(3-3t) \end{cases}$$

A cursory inspection, however, proves that in $\pi_1(X,a)$

(c) $\varphi(z)(f(x,y)) + f(z,xy) = f(z,x) + f(zx,y)$.

Now we must verify the relation

(a) $\varphi(x)(\varphi(y)(\sigma)) = f(x,y) + \varphi(xy)(\sigma) - f(x,y)$.

The left side is represented by

$$\begin{cases} P_x(3t) & 0 \leq t \leq \frac{1}{3} \\ xP_y(9t-3) & \frac{1}{3} \leq t \leq \frac{4}{9} \\ xy\sigma(9t-4) & \frac{4}{9} \leq t \leq \frac{5}{9} \\ xP_y(6-9t) & \frac{5}{9} \leq t \leq \frac{2}{3} \\ P_x(3-3t) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

while the right is represented by

$$\begin{cases} P_x(3t) \\ xP_y(3t-1) \\ P_{xy}(3-3t) \end{cases} + \begin{cases} P_{xy}(3t) \\ xy\sigma(3t-1) \\ P_{xy}(3-3t) \end{cases} + \begin{cases} P_{xy}(3t) \\ xP_y(2-3t) \\ P_x(3-3t). \end{cases}$$

But the identity is now obvious. Thus,

2.2. _Lemma_: _We have the group extension_

$$1 \to \pi_1(X,a) \to G \to F \to 1.$$

2.3. _Theorem_: _We may geometrically realize this extension as a group of homeomorphisms of the universal covering space of_ X^* _in such a way that_ (G,X^*) _covers the action_ (F,X) _under the map_ $\eta \times \nu : (G,X^*) \to (F,X)$ _where_ $\eta(\sigma,x) = x$ _and_ $\nu : X^* \to X$ _is the covering projection. The action_ (G,X^*) _is properly discontinuous if_ (F,X) _is properly discontinuous. Furthermore, there exists a canonical isomorphism of the isotropy groups,_ $F_{\nu(b)} \cong G_b$, $b \in X^*$.

Now $\pi_1(X,a)$ operates freely from the left as the group of covering transformations on X^*, the universal covering space. We wish to extend this to an action of G on X^*. To say $\pi_1(X,a)$ acts on X^* is to say that there is a homomorphism of $\pi_1(X,a)$ into the group of all homeomorphisms of X^*. We wish to extend this homomorphism to all of

G. To each $\sigma \in \pi_1(X,a)$ we associate a homeomorphism $h(\sigma) = \sigma_{\#} : X^* \to X^*$ as follows. A point $b \in X^*$ is represented by a path $p(t)$ issuing from a. Let $\sigma(t)$ be a loop at a representing σ. Then $\sigma_{\#}(b) \in X^*$ is the element represented by the path

$$\begin{cases} \sigma(2t) & 0 \le t \le \frac{1}{2} \\ p(2t-1) & \frac{1}{2} t \le 1 \end{cases}$$

To each $x \in F$ we now associate a homeomorphism $T_x : X^* \to X^*$. If b is represented by the path $p(t)$ then $T_x(b)$ is represented by

$$\begin{cases} P_x(2t) \\ xp(2t-1). \end{cases}$$

We must consider first the composition $(T_x \circ T_y)(b)$. This would be represented by the path

$$\begin{cases} P_x(2t) \\ xP_y(4t-2) \\ xyp(4t-3) \end{cases}$$

Compare this with $(f(x,y)_{\#} \circ T_{xy})(b)$ which is given by

$$\begin{cases} P_x(6t) \\ xP_y(6t-1) \\ P_{xy}(3-6t) \\ P_{xy}(4t-2) \\ xyp(4t-3). \end{cases}$$

By inspection we then see

$$T_x \circ T_y = f(x,y)_{\#} \circ T_{xy}.$$

We must also examine $(T_x \circ \sigma_{\#})(b)$, represented by

$$\begin{cases} P_x(2t) \\ x\sigma(4t-2) \\ xp(4t-3) \end{cases}$$

and $((\varphi(x)(\sigma))_{\#} \circ T_x)(b)$ which is represented by

$$\begin{cases} P_x(6t) \\ x\sigma(6t-1) \\ P_x(3-6t) \\ P_x(4t-2) \\ xp(4t-3) \end{cases}$$

and again we have

$$T_x \circ \sigma_{\#} = (\varphi(x)\sigma)_{\#} \circ T_x.$$

According to our opening lemma we can extend $\sigma \to \sigma_{\#}$ to a homomorphism of G into the group of all homeomorphisms of X^*. Hence G acts on X^*. If $\nu : X^* \to X$ is the projection map then $\nu((\sigma,x)_{\#}(b)) = x\nu(b)$ by definition. This completes the first part of the theorem.

We would now like to determine the isotropy group $G_b \subset G$. To each $x \in F_{\nu(b)}$ there corresponds a unique $\sigma \in \pi_1(X,a)$ with $\sigma_{\#} \circ T_x(b) = b$. Simply choose σ with $T_x(b) = (\sigma^{-1})_{\#}(b)$. This defines a function $F_{\nu(b)} \to G_b$ which is a 1-1 correspondence. Suppose (σ_1,x), (σ_2,y) lie in G_b, then

$$\begin{aligned} b &= (\sigma_1)_{\#} \circ T_x \circ (\sigma_2)_{\#} \circ T_y(b) \\ &= (\sigma_1)_{\#} \circ (\varphi(x)(\sigma_2))_{\#} \circ T_x \circ T_y(b) \\ &= (\sigma_1)_{\#} \circ (\varphi(x)(\sigma_2))_{\#} \circ f(x,y)_{\#} \circ T_{xy}. \end{aligned}$$

Hence $(\sigma_1 + \varphi(x)(\sigma_2) + f(x,y), xy \in G_b$ so that at each $b \in X^*$ we have a canonical isomorphism $F_{\nu(b)} \simeq G_b$.

Let us recall the definition of a properly discontinuous action (G,X^*). The discrete group G is said to operate properly discontinuously on X^* if

(a) If $b' \not\in Gb$, then there are neighborhoods U_b' and U_b with $U_b' \cap GU_b = \phi$.

(b) For each $b \in X^*$, the isotropy group G_b is finite.

(c) At each $b \in X^*$, there is a neighborhood U_b with $G_b U_b = U_b$, and such that if $U_b \cap gU_b = \phi$, then $g \in G_b$.

We shall now show that G acts properly discontinuously on X^* if the action (F,X) is properly discontinuous.

Let $b' \not\in G(b)$. Then $\nu(b') \not\in F(\nu(b))$. Choose $U_{\nu(b')}$ and $U_{\nu(b)}$ so that $U_{\nu(b')} \cap F(U_{\nu(b)}) = \phi$. Since X is semi 1-connected we may also choose $U_{\nu(b)}$ and $U_{\nu(b')}$ so that $i_* : \pi_1(U_{\nu(b)},\nu(b)) \to \pi_1(X,\nu(b))$ and $i_* : \pi_1(U_{\nu(b')},\nu(b')) \to \pi_1(X,\nu(b'))$ are trivial. Hence, $xU_{\nu(b)}$, and $U_{\nu(b')}$ are evenly covered. In fact, $\nu^{-1}(U_{\nu(b)}) = \pi_1(X,a)U_b$, where U_b is a lift of $U_{\nu(b)}$ to a neighborhood of b, and $G(U_b) = \nu^{-1}(F(U_{\nu(b)})$. Thus condition (a) is satisfied.

Let $U_{\nu(b)}$ be a neighborhood of $\nu(b)$ satisfying condition (c). Let V be a neighborhood of $\nu(b)$ which is evenly covered and choose $W_{\nu(b)} = \bigcap_{x \in F_{\nu(b)}} x(V \cap U_{\nu(b)})$. $W_{\nu(b)}$ is evenly covered and satisfies condition (c) as a neighborhood of $\nu(b)$. For $b \in X^*$, the lift of $W_{\nu(b)}$ to b, W_b, has the desired property for (c). Finally we point out that (G,X^*) is a group of covering transformations if and only if (F,X) is a group of covering transformations. This completes the proof of the theorem.

2.4. Corollary: If (F,X) is an (properly discontinuous) action of a finite (respectively; discrete) group on a finite dimensional aspherical space X then the following are equivalent.

(i) G _has no torsion._

(ii) F _acts freely on_ X.

(iii) G _acts on_ X^* _as a group of covering transformations._

Since X^* is contractible and finite dimensional any element of G with prime order has a fixed point. If G has no elements of finite order then $F_{\nu(b)} \cong G_b$ is trivial for all $b \in X^*$.

2.5. The converse problem for a finite F might be phrased as follows. Suppose we are given a group extension

$$1 \to \pi_1(X,a) \to G \to F \to 1$$

of $\pi_1(X,a)$ by a finite group. Can this extension be realized by the foregoing geometric construction? This can be done as follows. Consider the product of X^* with itself, Y, as the set of all functions

$$\chi : F \to X^*$$

recalling F is finite. We use the function $\varphi: F \to \mathrm{Aut}(\pi_1(X,a))$ to define an action $(\pi_1(X,a),Y)$. To each $\sigma \in \pi_1(X,a)$ associate a homeomorphism on Y by

$$\Sigma(\chi)(z) \equiv (\varphi(z)\sigma)_{\#}(\chi(z)).$$

It is readily seen $\sigma \to \Sigma$ is a homomorphism defining a left action of $\pi_1(X,a)$ on Y as a group of covering transformations. We use $f(x,y)$ to define a homeomorphism $T_x : Y \to Y$ by

$$(T_x(\chi))(z) \equiv f(z,x)_{\#}(\chi(zx)).$$

We wish to verify that this extends the homomorphism of $\pi_1(X,a)$. Now $(T_x \circ T_y)(\chi)(z) = (f(z,x)_{\#} \circ f(zx,y)_{\#})\chi(zxy)$. But $\varphi(z)(f(x,y)) + f(z,xy) = f(z,x) + f(zx,y)$, hence $(T_x \circ T_y)(\chi)(z) = (\varphi(z)f(x,y))_{\#} \circ f(z,xy)_{\#})(\chi)(z) = (F(x,y) \circ T_{xy})(\chi)(z)$. Again

$(T_x \circ \Sigma)(\chi)(z) = f(z,x)_\# \circ (\varphi(zx)(\sigma))_\# \chi(zx) = (f(z,x) + \varphi(zx)(\sigma))_\# \chi(zx) = (f(z,x) + \varphi(zx)(\sigma) - f(z,x) + f(z,x))_\# \chi(zx) = ((\varphi(z)(\varphi(x)\sigma))_\# \circ f(z,x)_\#)\chi(zx) = ((\varphi(z)(\varphi(x))\sigma)_\# \circ T_x)\chi(z)$. Thus by Lemma 2.1 the action of $\pi_1(X,a)$ on Y extends to an action of G on Y, and yields $(F, Y/\pi_1(X,a))$. Application of the preceding construction to $(F, Y/\pi_1(X,a))$ yields G with its action on Y.

2.6. Corollary: <u>If X is a finite dimensional aspherical space and $1 \to \pi_1(X,a) \to G \to F \to 1$ is an extension by a finite group for which G is torsionless then Y/G is an aspherical finite dimensional space with fundamental group G. In addition F acts freely on $Y/\pi_1(X,a)$.</u>

This corollary may be regarded as a geometric formulation of a theorem of J. P. Serre, see Swan [7]. The geometric dimension of a group is finite if and only if the algebraic dimension is finite. The corollary says that if G is a torsionless extension of a group π with finite algebraic (i.e., cohomological) dimension by a finite group, then G has finite algebraic dimension.

3. CENTERLESS FUNDAMENTAL GROUPS

To any action (G,X) of a finite group on a pathwise connected space there is canonically associated an abstract kernel $(\Psi, G, \pi_1(X))$; that is, a homomorphism $\Psi : G \to \mathrm{Out}(\pi)$ together with a group extension

$$1 \to \pi \xrightarrow{\alpha} L \xrightarrow{\beta} G \to 1$$

which realizes this abstract kernel, (§2.). We shall set the notation for this section. Choose any basepoint $x \in X$ and for each $g \in G$ select a path $P^g(t)$ joining x to gx. The corresponding automorphism g_* on $\pi_1(X,x)$ is the composition of the translation iso-

morphism $\pi_1(X,x) \simeq \pi_1(X,gx)$ with $P^g_\#(1-t) : \pi_1(X,gx) \simeq \pi_1(X,x)$. This g_* is unique up to an inner-automorphism and yields the abstract kernel $\Psi : G \to \mathrm{Out}(\pi)$. The extension cocycle $f : G \times G \to \pi$ assigns to (g,h) the element represented by the closed loop

$$\begin{cases} P^g(3t) & , 0 \leq t \leq \frac{1}{3} \\ gP^h(3t-1), & \frac{1}{3} \leq t \leq \frac{2}{3} \\ P^{gh}(3-3t), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

The extension group L acts from left on the universal covering space X^* and under the covering map $p : X^* \to X$ we have $p(\ell b) = \beta(\ell)p(b)$, all $\ell \in L$, $b \in X^*$.

3.1. Lemma: _If (G,M) is a p-group acting on an aspherical manifold with centerless fundamental group and if the associated abstract kernel $\Psi : G \to \mathrm{Out}(\pi)$ is trivial then (G,M) has a non-empty fixed point set with mod p cohomology isomorphic to $H^*(M;Z_p)$._

Since π is centerless and Ψ is trivial it follows from Theorem (8.8) (Homology, S. Maclane, p. 128) that $1 \to \pi \xrightarrow{\alpha} L \xrightarrow{\beta} G \to 1$ is equivalent to $1 \to \pi \to \pi \times G \to G \to 1$. In particular β admits a splitting homomorphism $\eta : G \to L$ whose image will be a p-group acting on a contractible manifold. The image has a fixed point, hence by projection so does the original (G,M). Thus we may choose the g_* so that $g \to g_*$ is a homomorphism $G \to \mathrm{Aut}(\pi)$. But Ψ is still trivial, hence $g_* \in \mathrm{Inn}(\pi)$ and π is still centerless so each g_* is conjugation by a unique $\alpha_g \in \pi$. Since $g \to \alpha_g$ is a homomorphism of a finite group into a torsionless group we see $g_* =$ identity for all $g \in G$. Accordingly, in the terminology of the appendix, $H^1(G;\pi) = \mathrm{Hom}(G;\pi) = \{0\}$ and $\Gamma_0 = \pi$, which completes the proof. (Alternatively, we now have $G \times \pi$ acting on M^* so that $T_g \circ (\sigma_\# b) = (g_*(\sigma))_\# \circ T_g(b) = \sigma_\# \circ T_g(b)$. Theorem 6.1 of [5] now implies our result. That is, $F(G,M^*) = E$ has the property that $\sigma_\#(E) = E$, for all $\sigma \in \pi_1(M)$.

Since E is path connected, $\nu(E) \subset F_a$, where F_a is the component of $G(G,M)$ containing $a = \nu(b)$, b is a basepoint of M^* in E and ν is the covering map. Finiteness of G and connectedness of E implies that $\nu(E) = F_a = E/\pi$, cf. [5; Lemma 3.4]. Since E is acyclic mod Z_p, $H^*(F_a;Z_p) \approx H^*(\pi;Z_p) \approx H^*(M;Z_p)$.)

3.2. <u>Theorem</u> (A. Borel): <u>If</u> (F,M) <u>is a finite group acting effectively on a closed aspherical manifold</u> M <u>with centerless fundamental group then the associated abstract kernel</u> $\Psi : F \to \mathrm{Out}(\pi)$ <u>is a monomorphism</u>.

Suppose this is false, then for some prime p there is a Sylow p-subgroup, G, of the kernel of Ψ. Applying the lemma we see that G has a fixed point set whose mod p cohomology is isomorphic to that of M, a closed manifold. If M is orientable then G acts trivially on M, contradicting effectiveness.

Suppose M is not orientable, then the action of G, G in the kernel of Ψ, can certainly be lifted to the oriented double covering M^d of M. Since G acts trivially on $\pi_1(M)$ it also acts trivially on $\pi_1(M^d)$ and once again effectiveness on M is contradicted.

3.3. Remarks: Actually in proof of 3.1 we only use the facts that M is a finite dimensional space whose universal covering is acyclic mod Z_p, and whose fundamental group is torsionless and centerless. In particular if M is a finite dimensional $K(\pi,1)$ with centerless π, then for any finite group G to act freely on M, the group G must be embedded in the group of homotopy classes of self homotopy equivalences of M. And, more generally, if G is a finite p-group acting without fixed points, then $\mathrm{Out}(\pi)$ must contain p-torsion.

Corollary 6.2 of [5] asserted that if (G,M^n) is an effective action of a finite group on a closed aspherical generalized manifold

over Z with a fixed point, then the homomorphism $G \to \mathrm{Aut}\ \pi$ was a monomorphism. No assumption on $\pi_1(M^n)$ being centerless is made there. The point of 3.1. is that the centerless assumption implies the existence of a fixed point set for each p-subgroup and consequently even yields an embedding of the p-Sylow subgroup of G into $\mathrm{Aut}\ \pi$.

Another interesting interpretation is that if (Z_p, M^n) is an effective free action on a closed aspherical manifold for which the generator induces a homeomorphism homotopic to the identity, then π must have a center. In particular, if $n = 3$, M is orientable, and π is "sufficiently large" then M is a Seifert manifold modulo the Poincare conjecture.

There are many examples of closed aspherical manifolds for which center $\pi = 1$. Moreover, A. Borel has exhibited examples from the theory of symmetric spaces and algebraic groups for which $\mathrm{Out}\ \pi$ is finite, [3].

4. AUTOMORPHISMS OF A SEMI-DIRECT PRODUCT

In this section we shall consider a group automorphism $\Phi : \pi \to \pi$ for which $I-\Phi_* : H_1(\pi,Q) \cong H_1(\pi,Q)$. We shall form the semi-direct product $L = \pi \circ Z$ with multiplication given by $(\alpha,n) \cdot (\beta,m) = (\alpha\Phi^n(\beta),n+m)$ and we shall determine the outer automorphism group, $\mathrm{Out}(L)$ and the center of L.

4.1. Lemma: The kernel of the natural homomorphism

$$L \to (L/[L,L]) \times_Z Q$$

is the subgroup $\pi \subset L$.

Proof: The short exact sequence

$$1 \to \pi \to L \to Z \to 0$$

is split by $\chi(n) = (e,n)$. We shall consider Q as a trivial L-module

and shall use the Lyndon spectral sequence to show that

$$\chi_* : H_1(Z;Q) \simeq H_1(L;Q).$$

There is then the spectral sequence $\{E^r_{s,t}, d_r\} \Rightarrow H_*(L;Q)$ with $E^2_{s,t} \simeq H_s(Z;H_t(\pi;Q))$. In particular,

$$E^2_{1,0} \simeq H_1(Z;H_0(\pi,Q)) = H_1(Z;Q)$$
$$E^2_{0,1} \simeq H_0(Z;H_1(\pi;Q)).$$

However, $I-\Phi_* : H_1(\pi;Q) \simeq H_1(\pi;Q)$, hence $E^2_{0,1} = 0$. Thus $\chi_* : H_1(Z;Q) \simeq H_1(L;Q)$ and the composite homomorphism

$$\pi \to H_1(\pi;Q) \to H_1(L;Q)$$

is trivial. Since any element $(\alpha,n) \in L$ may be written $(\alpha,0) \cdot \chi(n)$ the lemma follows.

This lemma proves that $\pi \subset L$ is a <u>characteristic</u> <u>subgroup</u>; that is, π is invariant under each automorphism of L. Thus every automorphism of L induces an automorphism of Z. We denote by $\mathrm{Aut}^+(L) \subset \mathrm{Aut}(L)$ the subgroup, which has index at most 2, of those automorphisms which induce the identity on Z.

If $\sigma \in \mathrm{Aut}^+(L)$, we may write

$$\sigma(\alpha,0) = c(\alpha),\ c \in \mathrm{Aut}(\pi)$$
$$\sigma(e,n) = (\varphi(n),n),\ \varphi : Z \to \pi \text{ a crossed-homomorphism.}$$

The second assertion follows since $(\varphi(n+m),n+m) = (\varphi(n),n) \cdot (\varphi(m),m) = (\varphi(n)\Phi^n\varphi(m),n+m)$. Since $(\alpha,n) = (\alpha,0)(e,n)$ we have $\sigma(\alpha,n) = (c(\alpha)\varphi(n),n)$. Now in addition we have

$$\sigma((e,n) \cdot (\alpha,0)) = \sigma(\Phi^n(\alpha),n) = (c\Phi^n(\alpha)\varphi(n),n).$$

But $\sigma(e,n) \cdot \sigma(\alpha,0) = \sigma((e,n)(\alpha,0))$ implies $(\varphi(n)\Phi^n c(\alpha),n) = (c\Phi^n(\alpha)\varphi(n),n)$. Thus we obtain the fundamental identity

$$\varphi(n)(\Phi^n(c(\alpha))) = (c(\Phi^n(\alpha)))\varphi(n).$$

To utilize this identity we now prove

4.2. Lemma: If $c \in \mathrm{Aut}(\pi)$ and if $\delta \in \pi$ is an element for which

$$\delta(\Phi(c(\alpha))) = (c\Phi(\alpha))\delta$$

then there is a unique crossed-homomorphism $\varphi : Z \to \pi$ such that $\varphi(1) = \delta$ and

$$\varphi(n)(\Phi^n(c(\alpha)) = (c(\Phi^n(\alpha)))\varphi(n).$$

Proof: Since Z is a free group there is a unique homomorphism $h : Z \to L$ with $h(1) = (\delta,1)$. If we write $h(n) = (\varphi(n),n)$ then $\varphi : Z \to \pi$ is the required crossed-homomorphism.

Observe that $e = \varphi(0) = \varphi(-1+1) = \varphi(-1)\Phi^{-1}(\delta)$ so that $\varphi(-1) = \Phi^{-1}(\delta^{-1})$. Now $\Phi^{-1}(\delta^{-1}) \cdot (\Phi^{-1}(c(\alpha))) = \Phi^{-1}(\delta^{-1}c(\Phi\Phi^{-1}(\alpha))\delta\delta^{-1}) = \Phi^{-1}(\delta^{-1}\delta\Phi c(\Phi^{-1}(\alpha)) \cdot \delta^{-1}) = c(\Phi^{-1}(\alpha)) \cdot \Phi^{-1}(\delta^{-1})$. The required identity may now be established by double induction since it was just verified for $n = -1$. Assume that the identity has been established for some n, $|n| > 0$. If $n > 0$ we write $\varphi(n+1) \cdot (\Phi^{n+1}(c(\alpha))) = \varphi(1)\Phi\varphi(n) \cdot \Phi^{n+1}(c(\alpha)) = \varphi(1)\Phi(\varphi(n)\Phi^n(c(\alpha))) = \varphi(1)\Phi(c\Phi^n(\alpha) \cdot \varphi(n)) = c(\Phi^{n+1}(\alpha))\ \varphi(1)\Phi(\varphi(n)) = c(\Phi^{n+1}(\alpha)) \cdot \varphi(n+1)$. If $n < 0$, we write $\varphi(n-1) \cdot (\Phi^{n-1}(c(\alpha))) = \varphi(-1)\Phi^{-1}(\varphi(n)\Phi^n(c(\alpha))) = \varphi(-1)\Phi^{-1}(c(\Phi^n(\alpha))\varphi(n)) = c(\Phi^{n-1}(\alpha))\varphi(-1)\Phi^{-1}(\varphi(n)) = c(\Phi^{n-1}(\alpha))\varphi(n-1)$. Thus the identity is established for all integers.

For each pair (c,δ), where $c \in \mathrm{Aut}(\pi)$ and $\delta \cdot \Phi(c(\alpha)) = (c\Phi(\alpha)) \cdot \delta$ we may define $\sigma \in \mathrm{Aut}^+(L)$ by

$$\sigma(\alpha,n) = (c(\alpha)\varphi(n),n).$$

We must show that σ is multiplicative. Thus, $\sigma(\alpha\Phi^n(\beta),n+m) = (c(\alpha)c(\Phi^n(\beta))\varphi(n)\Phi^n\varphi(m),n+m) = (c(\alpha)\varphi(n)\Phi^n(c(\beta)\varphi(m)),n+m) = (c(\alpha)\varphi(n),n) \circ (c(\beta)\varphi(m),m) = \sigma(\alpha,n) \cdot \sigma(\beta,m)$. The reader may show that both the kernel and cokernel of σ are trivial.

Composition is the group operation in $\mathrm{Aut}^+(L)$ and this corresponds to composition in $\mathrm{Aut}(\pi)$. However, $\sigma_1(\sigma_2(e,1)) = \sigma_1(\delta_2,1) = (c_1(\delta_2)\delta_1,1)$. Denoting $\mu(\delta) \in \mathrm{Inn}(\pi)$, conjugation by δ, we introduce the group G of all pairs (c,δ) for which $\mu(\delta) \circ \Phi \circ c = c \circ \Phi \in \mathrm{Aut}(\pi)$. The group operation in G is $(c_1,\delta_1) \cdot (c_2,\delta_2) = (c_1 \circ c_2, c_1(\delta_2)\delta_1)$. We have just shown that

4.3. Lemma: $\mathrm{Aut}^+(L) \simeq G$.

Clearly $\mathrm{Inn}(L) \subset \mathrm{Aut}^+(L)$ so let us describe the corresponding subgroup of G. For $(\alpha,n) \in L$ we have

$$(\alpha,n)(\beta,0)(\Phi^{-n}(\alpha^{-1}),-n) = (\alpha\Phi^n(\beta)\alpha^{-1},0)$$

$$(\alpha,n)(e,1)(\Phi^{-n}(\alpha^{-1}),-n) = (\alpha\Phi(\alpha^{-1}),1).$$

Thus to the inner-automorphism determined by (α,n) there corresponds $(\mu(\alpha) \circ \Phi^n, \alpha\Phi(\alpha^{-1}))$.

Let us denote by $K \subset \pi$ the center. We also denote by $C(\Phi) \subset \mathrm{Out}(\pi)$ the centralizer in the outer automorphism group of π and by $(\Phi) \subset \mathrm{Out}(\pi)$ the subgroup generated by Φ. Finally, since $\mathrm{Inn}(L) \subset \mathrm{Aut}^+(L)$ we have $\mathrm{Out}^+(L) \subset \mathrm{Out}(L)$ as a subgroup of index at most 2.

4.4. Theorem: <u>If the order of Φ in $\mathrm{Out}(\pi)$ is equal to its order in $\mathrm{Aut}(\pi)$ then there is a short exact sequence</u>

$$0 \to H_0(Z;K) \to \mathrm{Out}^+(L) \to C(\Phi)/(\Phi) \to 1.$$

Proof: Surely there is an epimorphism $G \to C(\Phi)$ and $(\mu(\alpha) \circ \Phi^n, \alpha\Phi(\alpha^{-1})) \to \Phi^n \in \mathrm{Out}(\pi)$. Thus we receive an epimorphism

$Out^+(L) \to C(\Phi)/(\Phi)$. A pair (c,δ) represents an element in the kernel of this epimorphism if and only if $c = \mu(\alpha) \circ \Phi^n$ for some $\alpha \in \pi$, $n \in Z$. But then $\mu(\delta) \circ \Phi \circ \mu(\alpha) \circ \Phi^n = \mu(\delta) \circ \mu(\Phi(\alpha)) \circ \Phi^{n+1} = \mu(\alpha) \circ \Phi^{n+1}$.

Thus $\mu(\delta) = \mu(\alpha\Phi(\alpha^{-1}))$ and so we may write $\delta = \beta \cdot \alpha\Phi(\alpha^{-1})$ for some $\beta \in K$ and $(c,\delta) = (id,\beta)(\mu(\alpha) \circ \Phi^n, \alpha\Phi(\alpha^{-1}))$. Thus $(c,\delta) = (id,\beta) \in Out^+(L)$ for some $\beta \in K$. The problem now is to determine when $(id,\beta) \in G$ is an inner-automorphism. If $(id,\beta) = (\mu(\alpha) \circ \Phi^n, \alpha\Phi(\alpha^{-1}))$ then $\Phi^n = I \in Out(\pi)$. Since the order of $\Phi \in Out(\pi)$ is equal to the order of $\Phi \in Aut(\pi)$ by hypothesis we have

$$\mu(\alpha) \circ \Phi^n = \mu(\alpha) = id \in Aut(\pi).$$

Thus $\alpha \in K$ and $\alpha\Phi(\alpha^{-1}) = \beta$. On K, $\alpha \to \alpha\Phi(\alpha^{-1})$ is an endomorphism with cokernel $H_0(Z;K)$. Hence we obtain

$$0 \to H_0(Z;K) \to Out^+(L) \to C(\Phi)/(\Phi) \to 1.$$

4.5. <u>Corollary</u>: <u>If, under the hypothesis of the theorem, the group</u> π <u>is abelian then the short exact sequence</u>

$$0 \to H_0(Z;\pi) \to Out^+(L) \to C(\Phi)/(\Phi) \to 1$$

<u>is split</u>.

<u>Proof</u>: Since π is abelian, $K = \pi$ and $Out(\pi) = Aut(\pi)$ and we may define $C(\Phi) \to G$ by $c \to (c,e)$, but then $\Phi^n \to (\Phi^n,e)$, which is just the inner-automorphism of L given by (e,n). Thus the splitting homomorphism $C(\Phi)/(\Phi) \to Out^+(L)$ is induced.

Now let us discuss the full group $Out(L)$. By analogy an element $\sigma \in Out^-(L)$ may be written $\sigma(\alpha,n) = (c(\alpha)\Psi(n),-n)$ where $\Psi(n+m) = \Psi(n)\Phi^{-n}\Psi(m)$ and

$$\Psi(n)(\Phi^{-n}(c(\alpha))) = (c\Phi^{n}(\alpha))\Psi(n).$$

Thus with $\delta = \Psi(1)$ we find $\mu(\delta) \circ \Phi^{-1} \circ c = c \circ \Phi$ so that in $\mathrm{Out}(\pi)$ we receive $\Phi^{-1} = c \circ \Phi \circ c^{-1}$. Thus if $N(\Phi) \subset \mathrm{Out}(\pi)$ is the subgroup of all elements for which $c \circ \Phi \circ c^{-1} = \Phi^{\pm 1}$ then $(\Phi) \subset C(\Phi) \subset N(\Phi)$. $N(\Phi)$ is the normalizer of (Φ) in $\mathrm{Out}(\pi)$.

4.6. Theorem: *If Φ has order > 2 under the hypothesis of the theorem 4.4. we again have a short exact sequence*

$$0 \to H_0(Z;K) \to \mathrm{Out}(L) \to N(\Phi)/(\Phi) \to 1,$$

which splits if π is abelian.

On the other hand, if $\Phi = \Phi^{-1}$ in $\mathrm{Aut}(\pi)$ then we may set up a 1-1 correspondence between $\mathrm{Out}^{-}(L)$ and $\mathrm{Out}^{+}(L)$. Namely, if $\sigma(\alpha,n) = (c(\alpha)\varphi(n),n) \in \mathrm{Aut}^{+}(L)$, then $\sigma^{1}(\alpha,n) = (c(\alpha)\varphi(n),-n) \in \mathrm{Aut}^{-}(L)$ since $\Phi^{-1} = \Phi$. This induces the correspondence between $\mathrm{Out}^{+}(L)$ and $\mathrm{Out}^{-}(L)$ (One may easily illustrate this case by the fundamental group L of the Klein bottle. Here L is the non-trivial semi-direct product of Z by Z and $\mathrm{Out}^{+}(L) \cong Z_2$, while $\mathrm{Out}^{-}(L) \cong Z_2$ and the full automorphism group $\mathrm{Out}(L) \cong Z_2 + Z_2$.)

In the remaining part of this section we shall determine when the semi-direct product, under the conditions at hand, possesses a center. We shall eventually prove

4.7. Theorem: *If the center of π is finitely generated then L has a trivial center if and only if*

(i) *Φ leaves no central element in π, other than the identity, fixed,*

(ii) *Φ has infinite order in $\mathrm{Out}\ \pi$.*

Proof: Suppose first that L is centerless. If there were a central element $\alpha \in K$, $\alpha \neq e$ and $\Phi(\alpha) = \alpha$, then $(\alpha,0) \in L$ is a central element because

$$(\alpha,0)(\beta,n) = (\alpha\beta,n)$$
$$(\beta,n)(\alpha,0) = (\beta\Phi^n(\alpha),n) = (\beta\alpha,n) = (\alpha\beta,n).$$

4.8. Lemma: Suppose that the order of Φ in $\mathrm{Out}(\pi)$ is equal to its order in $\mathrm{Aut}\,\pi$. If Φ leaves no central element of π other than the identity fixed, then the center of L is the subgroup of elements (e,n) with $\Phi^n = I \in \mathrm{Out}\,\pi$.

Proof: Suppose (α,n) is central in L then $(\alpha,n)(e,1) = (\alpha,n+1) = (e,1)(\alpha,n) = (\Phi(\alpha),n)$, hence $\alpha = \Phi(\alpha)$. On the other hand, $(\alpha,n)(\beta,0) = (\alpha\Phi^n(\beta),n) = (\beta,0)(\alpha,n) = (\beta\alpha,n)$ so that $\Phi^n(\beta) = \alpha^{-1}\beta\alpha$ and $\Phi^n = I \in \mathrm{Out}(\pi)$. But Φ is assumed to have the same order in $\mathrm{Aut}(\pi)$, so $\Phi^n = I, \beta = \alpha\beta\alpha^{-1}$ which implies $\alpha \in K$ and this together with $\Phi(\alpha) = \alpha$ means that $\alpha = e$. This completes the proof. In particular, if this mutual order is k, $0 < k < \infty$, then the center of L is the infinite cyclic subgroup of elements of the form (e,kn) and if Φ has infinite order, then L has trivial center.

Let us now suppose that in $\mathrm{Out}(\pi)$, Φ has finite order k, $0 < k < \infty$. There is then, for each integer n, a $\gamma \in \pi$, uniquely determined modulo k, such that $\Phi^{nk}(\beta) = \gamma^{-1}\beta\gamma$.

4.9. Lemma: $\gamma\Phi(\gamma^{-1}) \in K$.

Proof: Write $\Phi(\Phi^{nk}(\beta)) = \Phi^{nk+1}(\beta) = \Phi^{nk}(\Phi(\beta))$, which yields the identity $\Phi(\gamma^{-1})\Phi(\beta)\Phi(\gamma) = \gamma^{-1}\Phi(\beta)\gamma$. Since Φ is an automorphism it follows that $\gamma\Phi(\gamma^{-1})$ is central.

We may replace γ by $\gamma\alpha$, $\alpha \in K$. But then $\gamma\Phi(\gamma^{-1})$ is replaced by $\gamma\alpha\Phi(\alpha^{-1}\gamma^{-1}) = \gamma\Phi(\gamma^{-1})\alpha\Phi(\alpha^{-1})$. That is, the homology class

$c\ell(\gamma\Phi(\gamma^{-1})) \in H_0(Z;K)$ is well defined. We define $\chi : Z \to H_0(Z;K)$ by

$$\chi(n) = c\ell(\gamma\Phi(\gamma^{-1})).$$

4.10 <u>Lemma</u>: <u>The function</u> $\chi : Z \to H_0(Z;K)$ <u>is a homomorphism</u>.

<u>Proof</u>: Let $\Phi^k(\beta) \equiv \gamma^{-1}\beta\gamma$, for some $\gamma \in \pi$. $\Phi^k(\gamma) = \gamma^{-1}\gamma\gamma = \gamma$. Hence, by induction, $\Phi^{kn}(\beta) \equiv \gamma^{-n}\beta\gamma^n$, for each integer n, and of course, $\gamma^n\Phi(\gamma^{-n})$ lies in K also. We may write $\gamma^n\Phi(\gamma^{-n}) = \gamma^{n-1}\gamma\Phi(\gamma^{-1})\Phi(\gamma^{1-n}) = \gamma\Phi(\gamma^{-1})\gamma^{n-1}\Phi(\gamma^{1-n})$, $n > 0$ and $\gamma^n\Phi(\gamma^{-n}) = \gamma^{n+1}\gamma^{-1}\Phi(\gamma)\Phi(\gamma^{-1-n}) = \gamma^{-1}\Phi(\gamma)\gamma^{n+1}\Phi(\gamma^{-1-n})$, $n < 0$. So by induction we find that

$$c\ell(\gamma^n\Phi(\gamma^{-n})) = nc\ell(\gamma\Phi(\gamma^{-1}))$$

in $H_0(Z;K)$. Thus we have defined a homorphism $Z \to H_0(Z;K)$.

Now under the assumption that Φ leaves no element of K fixed we know that the endomorphism of K given by $\alpha \to \alpha\Phi(\alpha^{-1})$ must be a monomorphism. If K is assumed finitely generated then $H_0(Z;K)$, the cokernel of $K \to K$ by $\alpha \to \alpha\Phi(\alpha^{-1})$, must be a finite group. Hence there is an integer n, $0 < n < \infty$, with $c\ell(\gamma^n\Phi(\gamma^{-n})) = 0 \in H_0(Z;K)$. Thus, $\gamma^n\Phi(\gamma^{-n}) = \alpha\Phi(\alpha^{-1})$, for some $\alpha \in K$. Replace γ^n by $\rho = \gamma^n\alpha^{-1}$. Then $\Phi(\rho) = \rho$ and $\Phi^{nk}(\beta) \equiv \gamma^{-n}\beta\gamma^n = \alpha^{-1}\rho^{-1}\beta\rho\alpha = \rho^{-1}\beta\rho$.

We now claim that $(\rho,nk) \in L$ is a central element, for

$$(\beta,m)(\rho,nk) = (\beta\Phi^m(\rho),m+nk) = (\beta\rho,m+nk),$$
$$(\rho,nk)(\beta,m) = (\rho\Phi^{nk}(\beta),nk+m) = (\rho\rho^{-1}\beta\rho,nk+m)$$
$$= (\beta\rho,m+nk).$$

Since $nk = 0$, this is a non-trivial central element. We found this central element by assuming Φ has finite order in Out π. Thus if L is centerless, Φ has infinite order in Out π. We have already shown the converse and so this completes the proof of the Theorem 4.7.

4.11 Corollary: *If the center of* L *is trivial then* $\mathrm{Out}\,L$ *is finite if and only if* $C(\Phi)/(\Phi)$ *is finite.*

Proof: If the center of L is trivial then $\alpha \to \alpha\Phi(\alpha^{-1})$ is a monomorphism for all $\alpha \in K$. But then $H_0^{\Phi}(Z;K)$ is a finite group. Finally $\mathrm{Out}^+(L)$ is a normal subgroup of $\mathrm{Out}(L)$ with index at most 2.

There is an analogue of Theorem 4.4 when the order of Φ in $\mathrm{Out}(\pi)$ is not equal to its order in $\mathrm{Aut}(\pi)$. It is a corollary of above together with the proof of Theorem 4.4.

4.12. Corollary: *If we take* Φ *as in the beginning of this section, then there is an exact sequence*

$$Z \xrightarrow{\chi} H_0(Z;K) \to \mathrm{Out}^+(L) \to C(\Phi)/(\Phi) \to 1.$$

The image of χ is trivial if order Φ in $\mathrm{Out}(\pi)$ is equal to its order in $\mathrm{Aut}(\pi)$.

4.13. Remark: Much of the analysis that we used here in this section only depends upon working with a semi-direct product where the normal subgroup π is a characteristic subgroup of L. The quotient group need not necessarily be Z for obtaining a representation of $\mathrm{Aut}^+(L)$ as in 4.3. For example, if $L = Z^k \times \pi$ where π is centerless, then

$$0 \to \mathrm{Hom}(\pi,Z^k) \to \mathrm{Aut}(L) \overset{\leftarrow}{\to} \mathrm{Aut}(\pi) \times \mathrm{Aut}(Z^k) \to 1$$

is a split exact sequence, and

$$0 \to \mathrm{Hom}(\pi,Z^k) \to \mathrm{Out}(L) \overset{\leftarrow}{\to} \mathrm{Out}(\pi) \times \mathrm{Out}(Z^k) \to 1$$

is also a split exact sequence, $k \geq 0$. As an immediate application consider a closed orientable 2-manifold M of genus greater than 1. Let $L = Z \times \pi_1(M,x)$. It is known that the isotopy classes of homeomorphisms of $M \times S^1$ is isomorphic to $\mathrm{Out}\ \pi_1(M \times S^1)$, and hence isomorphic to $\mathrm{Out}(L)$.

5. THE ROLE OF $H^1(Z;K)$ in $\mathrm{Out}(L)$

5.1. Lemma: Let $\Phi : Z \to \mathrm{Aut}(K)$ be a homomorphism, where K is an abelian group. Then $H^1(Z;K)$ is canonically isomorphic to $H_0(Z;K)$.

Proof: Let $\mathrm{Hom}_\Phi(Z;K)$ be the abelian group of crossed-homomorphisms. Now $K \cong \mathrm{Hom}_\Phi(Z;K)$ since to each $\gamma \in K$ there corresponds a unique crossed-homomorphism $\varphi : Z \to K$ with $\varphi(n+m) = \varphi(n)\Phi^n(\varphi(m))$ and $\varphi(1) = \gamma$, (see 4.2). Furthermore, by uniqueness, φ is a principal crossed-homomorphism if and only if $\gamma = \alpha\Phi(\alpha^{-1})$. This correspondence establishes the isomorphism.

In the remainder of this section π is a torsionless group whose center, K, is finitely generated. Furthermore, $\Phi : \pi \to \pi$ is an automorphism for which

(i) $I - \Phi_* : H_1(\pi;Q) \cong H_1(\pi;Q)$

(ii) the order of Φ in $\mathrm{Out}(\pi)$ is infinite

(iii) Φ leaves no central element of π fixed other than the identity.

It follows now that $L = \pi \circ Z$ is a torsionless group with trivial center and there is a short exact sequence

$$0 \to H^1(Z;K) \to \mathrm{Out}^+(L) \to C(\Phi)/(\Phi) \to 1.$$

Because we assumed K is finitely generated and that the endomorphism of K given by $\gamma \to \gamma\Phi(\gamma^{-1})$ is a monomorphism, it follows that

$H^1(Z;K)$ is a finite group.

Now L is centerless and the embedding $H^1(Z;K) \to Out^+(L)$ may be regarded as an abstract kernel and hence there is a unique group extension $1 \to L \to G \to H^1(Z;K) \to 0$ which realizes this abstract kernel.

5.2. <u>Theorem</u>: <u>If the quotient</u> π/K <u>is also torsionless, the group extension which realizes</u> $H^1(Z;K) \to Out^+(L)$ <u>is torsion free</u>.

We must examine first the embedding $K \to Aut^+(L)$. To any $\gamma \in K$ there is a unique crossed-homomorphism $\varphi : Z \to K$ with

$$\varphi(1) = \gamma$$

$$\varphi(n+m) = \varphi(n)\Phi^m(\varphi(m)).$$

The corresponding automorphism $\Psi_\gamma \in Aut^+(L)$ is

$$\Psi_\gamma(\alpha,n) = (\alpha\varphi(n),n).$$

5.3. <u>Lemma</u>: <u>There is a</u> $k > 0$ and a $g = (\beta,m) \in L$ <u>for which</u> $(\mu(g) \circ \Psi_\gamma)^k = I \in Aut^+(L)$ <u>if and only if</u> $g = (\beta,0)$, $\beta^k \in K$ <u>and</u>

$$\gamma^k = \beta^{-k}\Phi(\beta^k).$$

Suppose first that such a $k > 0$ and $g \in L$ exist. Obviously $\Psi_\gamma^k = I \in Out^+(L)$. That is, for some $a = (\alpha,n)$, $\mu(a) = \Psi_\gamma^k$. But $\mu(a)$ corresponds to the pair $(\mu(\alpha) \circ \Phi^n, \alpha\Phi(\alpha^{-1}))$ while Ψ_γ corresponds to (I,γ). Since Φ has infinite order in $Out(\pi)$, $n = 0$ and $\alpha \in K$. Further, $\alpha\Phi(\alpha^{-1}) = \gamma^k$.

Now $(\mu(g) \circ \Psi_\gamma)^k = \mu(g) \circ \mu(\Psi_\gamma(g)) \dots \circ \mu(\Psi_\gamma^{k-1}(g)) \circ \Psi_\gamma^k$ $= \mu((\beta,m)\dots(\beta\varphi(m)^{k-1},m) \cdot (\alpha,0)) = I$. Since L is centerless, however,

$$(\beta,m)\dots(\beta\varphi(m)^{k-1},m)(\alpha,0) = (e,0) \in L.$$

Clearly $m = 0$ and $\beta^k\alpha = e$. Thus $\gamma^k = \beta^{-k}\Phi(\beta^k)$.

Conversely, if such a β exists then $(\mu(\beta,0) \circ \Psi_\gamma)^k =$

$I \in Aut^+(L)$.

Let us denote by $S \subset Aut^+(L)$ the subgroup generated by $Inn(L)$ and the image of $K \to Aut^+(L)$. Then S is the counter-image of the subgroup $H^1(Z;K)$ with respect to the quotient homomorphism $Aut^+(L) \to Out^+(L)$.

5.4. Lemma: *If π/K is torsionless, then S is a torsionless subgroup of $Aut^+(L)$.*

Proof: Any element of S can be written $\mu(g) \circ \Psi_\gamma$ for some $g \in L$, $\gamma \in K$, (since $\Psi_\gamma \circ \mu(g) = \mu(\Psi_\gamma(g)) \circ \Psi_\gamma$). If $\mu(g) \circ \Psi_\gamma$ has order k, then $g = (\beta,0)$ and $\beta^{-k}\Phi(\beta^k) = \gamma^k$. Our previous proof shows $\beta^k \in K$, hence $\beta \in K$ since π/K is torsionless. Now, however, using the fact that π is torsionless, $\beta^{-1}\Phi(\beta) = \gamma$. This implies Ψ_γ is the inner-automorphism corresponding to $(\beta^{-1},0)$ and hence $\mu(g) \circ \Psi_\gamma = I \in Aut^+(L)$.

Now we can prove the theorem stated. If in

$$1 \to L \xrightarrow{\nu} G \xrightarrow{\eta} H^1(Z;K) \to 0$$

there is an $h \in G$ of order k we see that $\eta(h) \in H^1(Z;K)$ has order exactly k also since L is torsionless. Thus the automorphism of L given by conjugation with h defines an element of S whose order is exactly k and under $Aut^+(L) \to Out^+(L)$ this automorphism corresponds to $\eta(h)$.

With this result we may construct a torsionless finite extension of L. If h has finite algebraic dimension, then G will also have the same finite dimension by the result of Serre (see 2.6 for a slightly weaker statement).

6. SOME MATRICES IN $GL(\ell,Z)$

A convenient source of examples to illustrate our general constructions is $GL(2,Z)$. For each integer $n > 0$ we introduce the matrix $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$, which has determinant -1, in $GL(2,Z)$.

6.1. Lemma: *There is no matrix* $M \in GL(2,Z)$ *for which*

$$\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix} M = -M \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}.$$

Proof: We write

$$\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c+a(n-1) & d+b(n-1) \\ c+an & d+bn \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix} = \begin{pmatrix} bn+a(n-1) & a+b \\ dn+c(n-1) & c+d \end{pmatrix}.$$

From $c + an = -dn - cn + c$ we have $a + c = -d$, while $-a - c + bn - b = -a - b$ yields $c = bn$. Now, however, $d + bn = -c - d = -bn - d$ implies that $d = -bn = -c$ and, therefore, $a = 0$. But then $c + a(n-1) = -bn - a(n-1)$ implies $c = 0$, $b = 0$ and $d = 0$ also.

Now, let $\eta : GL(2,Z) \to PGL(2,Z)$ denote the quotient by the subgroup of order 2 which $-I$ generates. We have shown that if $\eta(M)$ commutes with $\eta\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$ then M commutes with $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$ in $GL(2,Z)$.

6.2. Lemma: *There does not exist a matrix* $M \in GL(2,Z)$ *for which*

$$\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix} M = M \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}^{-1}.$$

Proof: Again we write

$$\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c+a(n-1) & d+b(n-1) \\ c+an & d+bn \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 1 \\ n & 1-n \end{pmatrix} = \begin{pmatrix} bn-a & b(1-n)+a \\ dn-c & d(1-n)+c \end{pmatrix}.$$

From the pair of equations

$$an - a + c = bn - a$$

$$an + c = dn - c$$

we find that $c = (d-b)n$. However, $bn + d = (1-n)d + c = d - nd + nd - bn$ shows that $b = 0$ since $n > 0$. From the relation $b(n-1) + d = b(1-n) + a$ we then obtain $a = d$. However, $c = nd$ so that finally $an + c = dn - c$ also shows $an = -c = -nd$ or $a = -d$. Hence the 0-matrix is the only possibility for M.

6.3. Lemma: *The centralizer of* $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$ *in* $GL(2,Z)$ *is the subgroup of matrices of the form* $\begin{pmatrix} a & b \\ bn & a+2b-bn \end{pmatrix}$ *with* $a(a+2b-bn) = \underline{+}1 + b_n^2$.

Proof: Immediate by direct calculation.

6.4. Lemma: *If* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\ GL(2,Z)$ *is a matrix for which* $ad > 0$, $bc > 0$ *then exactly one of the four matrices* $M, -M, M^{-1}, -M^{-1}$ *has all entries strictly positive.*

Proof: If $M \varepsilon SL(2,Z)$ then

$$M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$-M^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

while if $\det M = -1$

$$M^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

$$-M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Since $ad > 0$, $bc > 0$ all the possibilities are covered.

6.5. Lemma: If $k > 1$ there is no matrix $M \in GL(2,Z)$ for which $M^k = \pm\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$.

Proof: Suppose that such a matrix did exist. Since M commutes with M^k by Lemma 3 we can write $M = \begin{pmatrix} a & b \\ bn & a+2b-bn \end{pmatrix}$ with $a(a+2b-bn) = \pm 1 + b^2n$. We must eliminate some special cases when $n = 1$. These are $b = \pm 1$, $a = 0$; yielding the matrices,

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & \pm 1 \end{pmatrix} = \pm\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $b = \pm 1$, $a = \mp 1$, yielding,

$$\begin{pmatrix} \mp 1 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} = \pm\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Obviously these matrices are not solutions of $M^k = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with $k > 1$.

In general, thus we can assume $a(a+2b-bn) > 0$ and apply Lemma 4 to the equations

$$M^k = \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$$

$$M^{-k} = \begin{pmatrix} -1 & 1 \\ n & 1-n \end{pmatrix}$$

$$(-M)^k = (-I)^k\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$$

$$(-M^{-1})^k = (-I)^k\begin{pmatrix} -1 & 1 \\ 1 & 1-n \end{pmatrix}.$$

We see immediately that no one of the matrices $M, M^{-1}, -M, -M^{-1}$ can possibly have all terms positive since $k > 1$.

By observing M would still lie in the centralizer of $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$, the case $M^k = -\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$ is shown not to occur by a similar argument.

6.6. <u>Lemma</u>: <u>There is no element of finite order, other than</u> $-I$, <u>in the centralizer of</u> $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$.

<u>Proof</u>: Such an element would have the form $\begin{pmatrix} a & b \\ bn & a+2b-bn \end{pmatrix}$. If $b = 0$ we have $a^2 = 1$ corresponding to $\pm I$. When $n = 1$ we have exhibited all such matrices with $a(a+b) = 0$ and they are not of finite order. If $a(a+2b-bn) > 0$, then by Lemma 4 we see that the matrix cannot have finite order.

6.7. <u>Theorem</u>: <u>In</u> $GL(2,Z)$ <u>the centralizer of</u> $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$ <u>is the subgroup generated by this matrix together with</u> $-I$.

<u>Proof</u>: Let $C \subset PGL(2,Z)$ be the centralizer of $\eta\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$. According to Lemmas 6.1 and 6.6 there are no elements of finite order in C. Consider then the intersection $C \cap PSL(2,Z)$. This is a torsionless subgroup of the free product $Z_2 * Z_3$ and hence is itself free But every element in $C \cap PSL(2,Z)$ commutes with $\eta\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}^2$ and hence $C \cap PSL(2,Z)$ must be infinite cyclic. We have now $0 \to C \cap PSL(2,Z) \to C \to Z_2 \to 0$ and since C is torsionless it also follows that $C \subset PGL(2,Z)$ is also an infinite cyclic group.

We assert that $\eta\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$ must generate C, for if not then for some $k > 1$ there would be a matrix $M \in GL(2,Z)$ with $M^k = \pm\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$. Finally, by Lemma 6.1, $\eta^{-1}(C)$ is the centralizer of $\begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$ in $GL(2,Z)$.

Putting $\phi_n = \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$, $I - \phi_n = \begin{pmatrix} -n+2 & -1 \\ -n & 0 \end{pmatrix}$, and

$\det(I-\Phi_n) = -n$. Consequently, $H_0(Z, Z\oplus Z)$ = cokernel of the monomorphism $\alpha \to \alpha\Phi_n(\alpha^{-1})$ has order n, and is isomorphic to Z_n. The group $L(n) = (Z\oplus Z) \circ_{\Phi_n} Z$ is centerless by §4

6.8. <u>Corollary</u>: <u>There is a short exact sequence</u>

$$0 \to Z_n \to \mathrm{Out}(L(n)) \to Z_2 \to 0.$$

In fact, $\mathrm{Out}(L(n)) \cong Z_n \circ Z_2$ where Z_2 acts on Z_n by $\lambda \to -\lambda$.

If $n = 1$ then $\det(I-\Phi) = -1$ and hence $\mathrm{Out}(L(1)) \cong Z_2$. This represents the smallest group of outer automorphisms we have been able to construct.

We now turn to a matrix, $\Psi_b = \begin{pmatrix} -1 & 0 \\ -b & -1 \end{pmatrix}$, or simply Ψ, for each integer $b \neq 0$.

6.9. <u>Lemma</u>: $C(\Psi)$ <u>in</u> $GL(2,Z)$ <u>is the group generated by</u> $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ <u>and</u> $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ <u>in</u> $SL(2,Z)$ <u>and is isomorphic to</u> $Z\oplus Z_2$.

<u>Proof</u>: We first observe that $\Psi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ and hence $\Psi^n = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ nb & 1 \end{pmatrix}$.

Suppose that $(-I)\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}(-I)\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$. We then obtain, using the fact that $b \neq 0$,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ z & -1 \end{pmatrix}, \; z \in Z.$$

Since we may write

$$\begin{pmatrix} -1 & 0 \\ -r & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^r.$$

We see that $C(\Psi)$ is generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, for each b. In fact, the centralizer for Ψ and $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ coincide. Thus $C(\Psi) \cong Z\oplus Z_2$.

6.10. Lemma: The group $C(\Psi)/(\Psi) \cong Z_{2b}$.

Proof: There is an isomorphism $C(\Psi) \underset{\approx}{\to} Z + Z_2$ given by

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \to (y,0) \in Z \oplus Z_2, \text{ and}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \to (0,1) \in Z \oplus Z_2.$$

The group Ψ is generated by $\begin{pmatrix} -1 & 0 \\ -b & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, or in terms of $Z \oplus Z_2$ by $(b,1)$. The quotient group is Z_{2b}.

6.11. Lemma: $H_0(Z;Z\oplus Z) \approx Z_4$, b odd and $Z_2 \oplus Z_2$, for b even.

Proof: We must calculate the cokernel of the monomorphism $I - \Psi : Z\oplus Z \to Z\oplus Z$ given by the matrix $\begin{pmatrix} 2 & 0 \\ b & 2 \end{pmatrix}$. If we are given $\begin{pmatrix} x \\ y \end{pmatrix} \in Z\oplus Z$, its image is $\begin{pmatrix} 2x \\ bx+2y \end{pmatrix}$. The cokernel of this homomorphism is $Z_2 \oplus Z_2$ if b is even, but Z_4 if b is odd.

Let us form $L_b = (Z\oplus Z) \circ_{\Psi_b} Z$.

6.12. Theorem: L_b is a centerless group and there exist split exact sequences:

$$1 \to Z_2 \oplus Z_2 \to \mathrm{Out}^+(L_b) \to Z_{2b} \to 1, \ b \text{ even},$$

$$1 \to Z_4 \to \mathrm{Out}^+(L_b) \to Z_{2b} \to 1, \ b \text{ odd}.$$

Proof: Ψ is of infinite order in $GL(2,Z)$ and $\det(I-\Psi) = 4$ and Ψ leaves no element of $Z\oplus Z$ fixed except the identity.

6.13. Lemma: There is an exact sequence

$$1 \to C(\Psi)/(\Psi) \to N(\Psi)/(\Psi) \to Z_2 \to 0.$$

Proof: We must consider solutions for

$$\begin{pmatrix} -1 & 0 \\ -b & -1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} -1 & 0 \\ b & -1 \end{pmatrix}.$$

We obtain

$$\begin{pmatrix} -x, & -y \\ -bx-z, & -by-w \end{pmatrix} = \begin{pmatrix} -x+by, & -y \\ -z+bw, & -w \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 \\ z & 1 \end{pmatrix}.$$

These matrices do not lie in $C(\Psi)$ and hence there is an epimorphism of $N(\Psi)/(\Psi)$ onto Z_2 with kernel $C(\Psi)/(\Psi)$. It is easily seen that this epimorphism splits.

6.14. *Corollary*: $\mathrm{Out}(L_b) = (Z_2 \oplus Z_2) \circ N(\Psi)/(\Psi)$, b *even, and* $Z_4 \circ N(\Psi)/(\Psi)$, b *odd is a split extension of* $\mathrm{Out}^+(L_b)$ *by* Z_2.

The smallest example occurs when $b = 1$. $\mathrm{Out}(L_1)$ has order 16 and is non-abelian.

The next example will not yield a centerless group. Let $\tau = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ to obtain a matrix of order 6.

6.15. *Lemma*: *The centralizer* $C(\tau) \subset GL(2,Z)$ *is the subgroup generated by* τ.

Proof: From the computation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} b & b-a \\ d & d-c \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a+c & b+d \end{pmatrix}$$

we see that $C(\tau)$ is the subgroup of matrices of the form $\begin{pmatrix} a & b \\ -b & a-b \end{pmatrix}$ with $a^2 + b^2 - ab = \pm 1$. The condition on the determinant can be equivalently put as $a^2 + b^2 + (a-b)^2 = \pm 2$. Thus only $+2$ is possible and there are six solutions, corresponding to the elements in the subgroup generated by τ.

Let $\pi = (Z+Z) \circ_\tau Z$. Note that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ has determinant $+1$ so $H^1(Z;Z+Z) = 0$. Since $C(\tau)/(\tau) = \{1\}$ we conclude that $\mathrm{Out}^+(\pi)$ is trivial. Since τ has period 6, the center K is the subgroup of all elements $\{(0,0,6j)\}$. Now we observe that

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

and that $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^2 = I$. Thus $1 \to C(\tau) \to N(\tau) \to Z_2 \to 0$ is split exact. Furthermore,

6.16. <u>Proposition</u>: $\mathrm{Out}(\pi) \cong Z_2$ *and the generator of* $\mathrm{Out}(\pi)$ *is given by an automorphism of period* 2 *in* $\mathrm{Aut}(\pi)$. *Explicitly,*

$$\Phi(p,q,r) = (p+q, -q, -r).$$

Now we form $L = \pi \circ Z$ with respect to the automorphism Φ. On the center $K \subset \pi$, $\Phi(0,0,6j) = (0,0,-6j)$, thus $H^1(Z;K) \cong Z_2$ and the center of L is the infinite cyclic subgroup of elements $\{(e,2k)\}$. Furthermore $I - \Phi_* : H_1(\pi;Q) \cong H_1(\pi;Q)$. To see this we have only to recall that the homomorphism $Z \to \pi$ given by $r \to (0,0,r)$ induces $H_1(Z;Q) \cong H_1(\pi;Q)$ and on this image subgroup Φ is $r \to -r$.

Now $\mathrm{Out}(\pi) \cong Z_2$ so $C(\Phi)/(\Phi) = \{0\}$ since Φ generates $\mathrm{Out}(\pi)$. Thus we have

6.17. <u>Proposition</u>: $Z_2 \cong H^1(Z;K) = \mathrm{Out}^+(L)$, *and*

$$\mathrm{Out}(L) \simeq Z_2 \oplus H^1(Z;K) \simeq Z_2 \oplus Z_2.$$

We need only show

$$0 \to Z_2 \to \mathrm{Out}(L) \to Z_2 \to 0$$

splits. Since Φ has order 2 in $\mathrm{Aut}(\pi)$, however, the element of order 2 in $\mathrm{Aut}^{-}(L)$ is $(\alpha,n) \to (\alpha, -n)$.

In the next part of this section we shall examine a matrix in $SL(3,Z)$ that arises from our earlier considerations.

Let,

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \varepsilon\ SL(3,Z).$$

Then,

$$I - \Gamma = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and $\det(I-\Gamma) = -2$.

To calculate $C(\Gamma) \subseteq GL(3,Z)$, we have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

which yields

$$\begin{pmatrix} b & a+b & -c \\ c & d+e & -f \\ h & g+h & -1 \end{pmatrix} = \begin{pmatrix} d & e & f \\ a+d & b+e & c+f \\ -g & -h & -i \end{pmatrix}$$

This implies $b = d$, $h = -g$, $g + h = -h$ hence $g = h = 0$, $-c = f$ and $c + f = -f$ hence $c = f = 0$, and $a + b = e$. We get

$$\begin{pmatrix} a & b & 0 \\ b & a+b & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Since $\det = \pm 1$, we must have $i(a^2+ab-b^2) = \pm 1$. Consequently $i = \pm 1$ and $a^2 + ab - b^2 = \pm 1$. This is exactly what we have studied earlier for the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, where we found the set of matrices $\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}$ in $GL(2,Z)$ generated by $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus we have as generators of $C(\Gamma)$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence the group,

$$C(\Gamma)/(\Gamma) \cong Z_2 \oplus Z_2.$$

Next we claim that $H_0(Z;Z\oplus Z\oplus Z) = Z_2$ and that Γ leaves only $(0,0,0) \in Z\oplus Z\oplus Z$ fixed. This follows immediately from $\det(I-\Gamma) = 2$. As before $N(\Gamma) = C(\Gamma)$. Let $H = (Z\oplus Z) \circ_\Gamma Z$.

6.18. <u>Theorem</u>: <u>The group</u> H <u>is centerless and</u> $\mathrm{Out}(H)$ <u>is isomorphic to</u> $Z_2 \oplus Z_2 \oplus Z_2$.

<u>Proof</u>: We have already found the split exact sequence:

$$1 \to Z_2 \to \mathrm{Out}(H) \overset{\leftarrow}{\to} Z_2 \oplus Z_2 \to 0$$

Since $\mathrm{Aut}\, Z_2 = 1$, we obtain the desired result.

We shall now give a procedure, similar to that just treated, for finding interesting matrices $\Phi \in GL(\ell,Z)$ for any positive ℓ. The matrices which we will describe first lie in $GL(2\ell,Z)$.

Let us rename $\Phi_n = \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix}$, x_n, $n > 0$. Let y denote an arbitrary 2×2 matrix with integer entries and determinant not necessarily different from 0.

6.19. Lemma: There are no solutions other than $y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for:

$$y \cdot x_n = x_m \cdot y, \ n \neq m$$

$$y \cdot x_n = x_n^{-1} \cdot y,$$

$$y \cdot x_n = x_m^{-1} \cdot y, \text{ and}$$

$$x_n \cdot y = y \cdot x_m^{-1}.$$

Proof:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix} = \begin{pmatrix} m-1 & 1 \\ m & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ yields}$$

$$\begin{pmatrix} a(n-1)+bn & a+b \\ c(n-1)+dn & c+d \end{pmatrix} = \begin{pmatrix} a(m-1)+c & b(m-1)+d \\ am+c & bm+d \end{pmatrix}.$$

We have $b = (c-a)+(d-b)$, $a = (n-1)(c-a)+n(d-b)$, hence, $c = a+(c-a) = n((c-a)+(d-b)) = nb$. Therefore, $a(n-1)+c = a(m-1)+c$ and $a = 0$ since $n \neq m$. Also as $c = nb = mb$, $c = b = 0$, hence $d = 0$.

Consider now,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n-1 & 1 \\ n & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ m & -(m-1) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a(n-1)+bn & a+b \\ c(n-1)+dn & c+d \end{pmatrix} = \begin{pmatrix} -a+c & -b+d \\ ma-(m-1)c & mb-(m-1)d \end{pmatrix}.$$

We have $c = (a+b)n$ and $m(b-d) = c = -(a+b)m$. Hence, $(a+b)n = -m(a+b)$, and since n and $m > 0$, $a+b = 0$. Therefore $c = 0$, $b = -a$ and $d = b$, $dn = ma = -mb$ and as $n = -m$, $b = d = 0$, hence $a = 0$.

Finally, we show $x_m \cdot y = y \cdot x_n^{-1}$ has only trivial solutions.

$$\begin{pmatrix} a(m-1)+c & b(m-1)+d \\ am+c & bm+d \end{pmatrix} = \begin{pmatrix} -a+bn & a-b(n-1) \\ -c+dn & c-d(n-1) \end{pmatrix},$$

$$\left.\begin{array}{l} am-bn = -c \\ am-dn = -2c \end{array}\right\} \Rightarrow \begin{array}{l} c = (d-b)n \quad \text{and,} \\ c = dn+bm \end{array}$$

hence, $0 = b(n+m)$ or $b = 0$, $d = a$, $am = -c = -nd = -na$, hence $a = d = c = 0$.

Let $n_1, n_2, \ldots, n_\ell$ be distinct positive integers. Form the matrix $\Phi(n_1, \ldots, n_\ell) \in GL(2\ell, Z)$ by considering blocks of (2×2)-matrices

$$\Phi(n_1, \ldots, n_\ell) = \begin{pmatrix} x_{n_1} & 0 & 0 & \cdots & 0 \\ 0 & x_{n_2} & 0 & \cdots & 0 \\ 0 & 0 & x_{n_3} & & \cdot \\ 0 & & & & \cdot \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & x_{n_\ell} \end{pmatrix}$$

We wish to compute the centralizer $C(\Phi)$ and $C(\Phi)/(\Phi)$ as well as the normalizer. Let c be an arbitrary matrix in $GL(2\ell, Z)$. We wish to determine the solutions to $c\Phi = \Phi c$. Write c in (2×2)-blocks and denote the (i,j)th block by c_{ij}.

$$(c\Phi)_{(i,j)} = \Sigma_k c_{ik}\Phi_{k,j} = c_{ij}\Phi_{j,j} = c_{ij}x_{n_j}$$

$$(\Phi \cdot c)_{i,j} = \Sigma_k \Phi_{ik}c_{kj} = \Phi_{ii}c_{ij} = x_{n_i}c_{i_j}.$$

Thus, if $c\Phi = \Phi c$ we must have

(i) $\qquad c_{ij}x_{n_j} = x_{n_i}c_{i_j}, \; i \neq j$

(ii) $\qquad c_{ii}x_{n_i} = x_{n_i}c_{ii}, \; i = j.$

By Lemma 6.19, we have seen that there are no solutions for (i) other than the trivial $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ solution. Thus, each diagonal c_{ii} block must have determinant ± 1. But, then, we have found the solutions for (ii) in 6.7. They are the matrices generated by x_{n_i} and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Let us denote by $-I_i$, the diagonal matrix in $GL(2\ell, Z)$ whose entries are all 1 except for (i,i)-th block which is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and by X_{n_i} the diagonal block matrix whose (i,i)-th block is x_{n_i} and whose diagonal blocks are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $C(\Phi(n_1,\ldots,n_\ell))$ is generated by $X_{n_1}, X_{n_2}, \ldots, X_{n_\ell}, -I_n, \ldots, -I_\ell$. This group is obviously $(Z \oplus Z_2) \oplus \ldots \oplus (Z \oplus Z_2) = (Z \oplus Z_2)^\ell$. The infinite cyclic subgroup generated by Φ is the diagonal subgroup of Z^ℓ. Thus $C(\Phi)/(\Phi) = Z^{\ell-1} + (Z_2)^\ell$, $\ell \geq 1$.

We now wish to show that the normalizer, $N(\Phi)$, is the same as the centralizer. We must find all solutions $c \in GL(2\ell, Z)$ of the equation

$$c\Phi = \Phi^{-1}c.$$

We have $\Phi^{-1}(x_{n_1}, x_{n_2}, \ldots, x_{n_\ell}) = \Phi(x_{n_1}^{-1}, \ldots, x_{n_\ell}^{-1})$.

$$(c\Phi)_{i,j} = \Sigma_k c_{ik}\Phi_{kj} = c_{ij}\Phi_{jj} = c_{ij}x_{n_j}$$

$$(\Phi^{-1}c)_{i,j} = \Sigma_k \Phi_{ik}^{-1}c_{kj} = \Phi_{ii}^{-1}c_{ij} = x_{n_i}^{-1}c_{ij}.$$

Again by Lemma 6.19 we see that each $c_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, $N(\Phi) = C(\Phi)$.

We now wish to determine the cokernel of $\alpha \to (I-\Phi)(\alpha)$, where $\alpha \in Z^{2\ell}$. The matrix $(I-\Phi)$ is a block diagonal matrix with the (i,i)-th block $\begin{pmatrix} 2-n_i & -1 \\ -n_i & 0 \end{pmatrix}$ and with determinant $(-1)^\ell \Pi_{i=1}^{\ell} n_i$. The cokernel splits up along the blocks and we obtain $Z_{n_1} \oplus Z_{n_2} \oplus \ldots \oplus Z_{n_\ell}$. The action of $1 \times 1 \times \ldots \times 1 \times Z_2 \times 1 \ldots \times 1$, with Z_2 in the i-th position on $Z_{n_1} \oplus \ldots \oplus Z_{n_i} \oplus \ldots \oplus Z_{n_\ell}$ is trivial on all factors except the i-th factor where it sends $\lambda \to -\lambda$. The action of the free part is trivial. The extension so defined is $Z^{\ell-1} \oplus D_{n_1} \oplus \ldots \oplus D_{n_\ell}$,

where the D_{n_i} are dihedral groups: $0 \to Z_{n_i} \to D_{n_i} \overset{\leftarrow}{\to} Z_2 \to 0$. We define the semi-direct product

$$L(n_1,n_2,\ldots,n_\ell) = Z^{2\ell} \circ_{\Phi(n_1,n_2,\ldots,n_\ell)} Z,$$

and have shown

6.20. Theorem: *The sequence* $0 \to Z_{n_1} \oplus \ldots \oplus Z_{n_\ell} \to \text{Out } L(n_1,\ldots,n_\ell) \to Z^{\ell-1} \oplus (Z_2)^\ell \to 0$ *is exact and splits. In fact,* $\text{Out } L(n_1,\ldots,n_\ell) \cong Z^{\ell-1} \oplus D_{n_1} \oplus \ldots \oplus D_{n_\ell}$, *where* D_{n_i} *are dihedral groups* $0 \to Z_{n_i} \to D_{n_i} \overset{\leftarrow}{\to} Z_2 \to 0$. *Furthermore, the group* L *is centerless.*

Since $\Phi \in GL(2\ell,Z)$ we may find an element of $GL(2\ell+1,Z)$ by adding the (1×1)-matrix block (-1) to Φ as we did earlier in 6.18. Also, if we wish our element in $GL(\ell,Z)$ to have positive determinant, we may add to or replace the last (2×2) matrix block by $\begin{pmatrix}-1 & 0\\ -1 & -1\end{pmatrix}$, $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$. We shall now examine how these changes slightly alter the computation above.

In $c\Phi = \Phi c$ where $\Phi \in GL(2\ell+1,Z)$, $\Phi = \Phi(x_{n_1},x_{n_2},\ldots,x_{n_\ell},-1)$ we obtain in addition to the equations (i) and (ii) of the previous discussion additional equations from the bottom row and the last column. Using the fact that $x_n = \begin{pmatrix}n-1 & 1\\ n & 1\end{pmatrix}$, one easily checks that c has the previous form for the $2\ell \times 2\ell$ submatrix and ± 1 in the $(2\ell+1,2\ell+1)$-st entry with 0's otherwise in the last row and last column. That is,

$$C(\Phi(x_{n_1},x_{n_2},\ldots,x_{n_\ell},-1)) = C(\Phi(x_{n_1},x_{n_2},\ldots x_{n_\ell}))\oplus\begin{pmatrix}1 & 0 & \cdot & \cdot & \cdot & 0\\ 0 & 1 & & & & \cdot\\ \cdot & & & & & \cdot\\ \cdot & & & & & \cdot\\ \cdot & & & & & \cdot\\ \cdot & & & & 1 & 0\\ 0 & \cdot & \cdot & \cdot & 0 & -1\end{pmatrix}$$

$= (Z \oplus Z_2)^\ell + Z_2$. We also claim that it is easy to check that $N(\Phi) = C(\Phi)$ and so

6.21. <u>Corollary</u>: L <u>is centerless, and</u>

$$\mathrm{Out}\, L(n_1,n_2,\ldots,n_\ell,-1) = Z^{\ell-1} \oplus D_{n_1} \oplus \ldots \oplus D_{n_\ell} \oplus Z_2 \oplus Z_2.$$

We now wish to add one of the (2×2) matrix blocks

(a_1) $\quad \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$

(a_2) $\quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, or

(a_3) $\quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

to the matrix $\Phi(n_1,\ldots,n_\ell)$ to get $\Phi(n_1,n_2,\ldots,n_\ell,a_j) \in GL(2(\ell+1),Z)$. In order to compute $\mathrm{Out}\, L$ we need to have the analogue of Lemma 6.19 for the matrices a_j above. In fact, by direct computation,

$$a_j \cdot y = y \cdot x_m$$

$$y \cdot a_j = x_m \cdot y$$

have only the trivial $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ solution for y a (2×2)-matrix with integer entries.

6.22. <u>Corollary</u>: <u>Each of the groups</u> $L(n_1,n_2,\ldots,n_\ell,a_j)$ <u>is centerless</u>. $\mathrm{Out}\, L(n_1,n_2,\ldots,n_\ell,a_j) \cong \mathrm{Out}\, L(n_1,\ldots,n_\ell) \times A_j$, <u>where</u> $A_1 = Z_4 \circ (Z_2 \times Z)$, $A_2 = Z_6$ <u>and</u> $A_3 = (Z_2 \oplus Z_2) \times GL(2,Z)$.

To show that $\mathrm{Out}\, L = \mathrm{Out}^+ L$ one need only check that

$$y \cdot x_m = a_j^{-1} \cdot y$$

$$y \cdot a_j = x_m^{-1} \cdot y, \text{ where } j = 2 \text{ and } x_m = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

have only trivial solutions $y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The exception $y \cdot a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot y$ has solution $\begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$. However, the trivial solutions to the first equation guarantee that the first column of the matrix c

has only 0 entries. Hence, there is no solution c in $GL(2(\ell+1),Z)$ of the equation $c\Phi = \Phi^{-1}c$. We re-emphasize that we have introduced these last complications so that one can, if one wishes, find desirable $\Phi \in GL(\ell,Z)$ with positive determinant.

7. TOPOLOGICAL EXAMPLES

7.1. Let Y be a topological space and $\Phi : (Y,y_0) \to (Y,y_0)$ be a homeomorphism. On $R^1 \times Y$ we introduce an action of Z as a group of covering transformations by $n(r,y) = (r-n, \Phi^n(y))$. Then $X(\Phi) = R \times_Z Y$ is the quotient, and a point in $X(\Phi)$ is written $((r,y))$. There is the map $\nu : X(\Phi) \to S^1$ given by $\nu((r,y)) = \exp(2\pi ir)$. This is a fibre map with fibre Y and structure group Z. Hence, $X(\Phi)$ is a closed aspherical manifold if Y is. The cross-section $\chi : S^1 \to X(\Phi)$ is given by $\chi(\exp(2\pi ir)) = ((r,y_0))$, which is well defined since y_0 is fixed. For the preferred base point of $X(\Phi)$ we use $x_0 = ((0,y_0))$. From the map $(Y,y_0) \to (X(\Phi),x_0)$ given by $y \to ((0,y))$ we then obtain a short exact sequence $1 \to \pi_1(Y,y_0) \to \pi_1(X,x_0) \to \pi_1(S^1) \to 0$ which is split by χ_*. Thus we may canonically identify $\pi_1(X(\Phi),x_0)$ with the semi-direct product $\pi_1(Y,y_0) \circ Z$ formed with respect to $\Phi_* : \pi_1(Y,y_0) \to \pi_1(Y,y_0)$. Let $K \subset \pi_1(Y,y_0)$ be the center. Let $N(\Phi_*) \subset Out(\pi_1(Y,y_0))$ be the normalizer of the cyclic subgroup, (Φ_*), generated by $\Phi_* \in Out(\pi_1(Y,y_0))$. If Φ_* satisfies the hypothesis of 4.6 we have the short exact sequence:

$$0 \to H_0^{\Phi_*}(Z;K) \to Out(\pi_1(X(\Phi),x_0)) \to N(\Phi_*)/(\Phi_*) \to 1$$

which splits if $\pi_1(Y,y_0)$ is abelian. Furthermore, 4.7 guarantees when $\pi_1(X(\Phi),x_0)$ is centerless and 4.8 will compute its center otherwise.

Let us apply this to the algebraic examples constructed in §6.

Let $\Phi_n : T^2 \to T^2$ be the automorphism of the 2-dimensional torus given by

$$\Phi_n(z_1,z_2) = (z_1^{n-1}z_2,\ z_1^n z_2)$$

and let

$$M(n) = X(\Phi_n) = R \times_Z T^2,$$

where $((r,z_1,z_2)) \sim ((r-1,\ z_1^{n-1}z_2,\ z_1^n z_2))$ in $M(n)$. This closed aspherical 3-manifold has centerless fundamental group isomorphic to $L(n)$ of 6.1 - 6.8.

7.2. *Corollary: The manifold* $M(1)$, $\left(\Phi_* = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right)$, *admits an involution, but no periodic maps of larger period. Furthermore, every involution on* $M(1)$ *has exactly two disjoint circles as fixed point set.*

For the first part of the lemma we apply Borel's Theorem (§3) and the fact that $Out(L(1)) \simeq Z_2$. For the second part we recall that since $L(1)$ is centerless there is an extension

$$1 \to L(1) \to G \to Out(L(1)) \to 0$$

which is unique to within a Baer equivalence. But then $G \simeq L(1) \circ Z_2$ since $Out(L(1))$ is generated by the image of an automorphism in $Aut(L(1))$ which has period 2. By the realization procedure (§2) then any involution on $M(1)$ is covered by some action of $L(1) \circ Z_2$ on the contractible universal covering space of $M(1)$.

Finally to see that the fixed point set consists of exactly 2 circles one may apply the results of the Appendix.

7.3. *Corollary: For* $M(n)$, *all the elements of* $Out\ L(n) \simeq Z_n \circ Z_2$ *may be realized by periodic homeomorphisms.*

Proof: For all $n > 0$, the involution $((r,z_1,z_2)) \to ((r,z_1^{-1},z_2^{-1}))$

is well defined and has fixed points. Simply observe that $((r-1, z_1^{n-1}z_2, z_1^n z_2)) \to ((r-1, z_1^{1-n}z_2^{-1}, z_1^{-n}z_2^{-1}))$ is equivalent to the above. Similarly, let λ be a primitive n-th root of unity, $n > 1$. Define a free action of Z_n by $((r,z_1,z_2)) \to ((r,z_1\lambda, z_2\lambda^2))$. One may use these two actions to define the others. Observe that all these manifolds are boundaries in the non-oriented sense.

7.4. Corollary: *Let $n > 2$ and T be a non-trivial periodic homeomorphism on $M(n)$. Then the period k divides $2n$. If k is odd then the group generated by T acts freely.*

Proof: For $n > 2$, Out $L(n) \cong Z_n \circ Z_2$ and the action of Z_2 on Z_n is by $\lambda \to -\lambda$, by 6.8. Suppose T is a periodic homeomorphism of period k. Then $Z_k \subseteq Z_n \circ Z_2$. If $\gamma \in Z_k$ and $\gamma = (\lambda,1)$ then $\gamma\cdot\gamma = (\lambda+1_*(\lambda), 1+1) = (\lambda-\lambda,0) = (0,0)$. Hence γ is of order 2. Thus $Z_k \subseteq Z_n$, if k is odd. We have seen from §5 that no element in the image of $Z_n \to$ Out $L(n)$ can be represented by a periodic automorphism in Aut $L(n)$. Hence, the group generated by T must be free.

Using the action of Z_2 with fixed points mentioned above we see that $M(n)/Z_2$ is the non-oriented 2-sphere bundle over the circle because each torus fibre is invariant and the quotient space of the involution $(z_1,z_2) \to (z_1^{-1},z_2^{-1})$ is the 2-sphere.

Perhaps any smooth involution is equivalent to the one we have exhibited. However, one can certainly find an infinite number of non-conjugate topological involutions by simply choosing an invariant 3 ball around a point in the fixed set and altering the linear involution inside the ball and keeping it unchanged on the boundary. One does this so that the fixed point set is no longer tamely embedded inside this ball. This procedure was first described by D. Montgomery and R. H. Bing.

We finally give a presentation for

$$\pi_1(M(n)) = \{x,x_1,x_2 | x_1x_2x_1^{-1}x_2^{-1}=1, xx_1x^{-1}=x_1^{n-1}x_2^n, xx_2x^{-1}=x_1x_2\},$$

hence $H_1(M(n);Z) \cong Z \oplus Z_n$.

7.5. The groups $L_b = (Z\ Z) \circ_{\Psi_b} Z$ of 6.9 - 6.14 may be geometrically realized as the fundamental groups of the total space of S^1-bundles over the Klein bottle with structure group $O(2)$.

We fix an integer $b \neq 0$ and introduce an automorphism on T^2 by

$$\Psi(z_1,z_2) = (z_1^{-1}, z_1^{-b}z_2^{-1}).$$

At the identity in T^2 the automorphism Ψ_* induced on the fundamental group is given by the matrix $\begin{pmatrix} -1 & 0 \\ -b & -1 \end{pmatrix} \in GL(2,Z)$. We introduce the closed orientable aspherical 3-manifold $M(b) = R \times_Z T^2$ obtained as before by identifying for each integer k

$$(r,z_1,z_2) \sim (r-k, z_1^{(-1)^k}, z_1^{(-1)^k bk} z_2^{(-1)^k}).$$

We can also fiber $M(b)$ over the Klein bottle, K, with fibre S^1. We regard K as obtained from T^2 by the identification $(t_1,t_2) \sim (-t_1, t_2^{-1})$. A point in K is denoted by $< t_1,t_2 >$. The map $p : M \to K$ is $p((r,z_1,z_2)) = < \exp(\pi ir), z_1 >$. We must show p is well defined. Now

$$((r-k, \Psi^k(z_1,z_2))) = ((r-k, z_1^{(-1)^k}, z_1^{(-1)^k kb} z_2^{(-1)^k})) = ((r,z_1,z_2)),$$

however $(-1)^k \exp(\pi ir) = \exp(\pi i(r-k))$ and $<(-1)^k\exp(\pi ir), z_1^{(-1)^k}> = < \exp(\pi ir), z_1> \in K$. Thus p is well defined.

Suppose $p((r,z_1,z_2)) = p((r_1,\mathfrak{z}_1,\mathfrak{z}_2))$. Then either

$$\begin{cases} \exp \pi i r = \exp \pi i r_1 \\ z_1 = \mathfrak{z}_1 \end{cases}$$

or

$$\begin{cases} -\exp \pi i r = \exp \pi i r_1 \\ z_1 = \mathfrak{z}_1^{-1} \end{cases}$$

In either case $r_1 = r + k$ and $z_1 = \mathfrak{z}_1^{(-1)^k}$. But $((r+k,\mathfrak{z}_1,\mathfrak{z}_2)) = ((r,\phi^k(\mathfrak{z}_1,\mathfrak{z}_2))) = ((r,\mathfrak{z}_1^{(-1)^k},\mathfrak{z}_1^{(-1)^k bk}\mathfrak{z}_2^{(-1)^k}))$ $= ((r,z_1,z_1^{bk}\mathfrak{z}_2^{(-1)^k}))$. Therefore, the map $S^1 \to M$ given by $t \to ((r,z_1,t))$ maps S^1 homeomorphically onto the fibre over $< \exp \pi i r,\ z_1 >$. The structure group of the bundle $M \to K$ is $O(2)$. There is a bundle $\nu_1 : K \to S^1$ with fibre S^1 and structure group Z_2 given by $< t_1,t_2 > \to t_1^2$. The diagram

$$\begin{tikzcd} & M \arrow[dl, "\nu"'] \arrow[d, "p"] \\ S^1 & K \arrow[l, "\nu_1"] \end{tikzcd}$$

commutes. In fact $M(b)$ is a Seifert manifold of type (O,n,II) and it has a double covering a principal circle bundle over the torus with Euler class $2b$.

We have calculated $\mathrm{Out}(\pi_1(M(b)))$ in 6.9 - 6.14, and shown that $\pi_1(M(b))$ is centerless. This gives us another class of manifolds which allows only a few finite groups to act. Since these manifolds are orientable $\mathrm{Out}(\pi_1(M(b)))$ is isomorphic to the group of isotopy classes of homeomorphisms of $M(b)$ onto itself. We further mention that the groups Z_4, b odd and Z_2+Z_2 can be realized as free actions on $M(b)$.

7.6. The group π of 6.15 can be realized as the fundamental group of an orientable closed flat 3-manifold as follows. On T^2 let

τ be the automorphism $\tau(z_1,z_2) = (z_2^{-1},z_1z_2)$, which has period 6. Let $\lambda = \exp 2\pi i/6$ and on $S^1 \times T^2$ introduce the identification $(t,z_1,z_2) \sim (t\lambda^{-1},z_2^{-1},z_1z_2)$. This yields $Y = (S^1 \times T^2)/Z_6$, which is a closed aspherical 3-manifold, acted on by S^1, whose fundamental group is π. Denoting by $<t,z_1,z_2>$ a point in Y we introduce a diffeomorphism of period 2

$$\Phi< t,z_1,z_2> = < t^{-1},z_1z_2,z_2^{-1} >$$

which corresponds to the automorphism Φ on π. Again on $S^1 \times Y$ we make the identification $(t_1,< t_2,z_1,z_2>) \sim (-t_1,< t_2^{-1},z_1z_2,z_2^{-1}>)$ to obtain the closed aspherical 4-manifold $X = (S^1 \times Y)/Z_2$, whose fundamental group is L (of 6.17) and which is acted on by S^1 effectively. The group $\mathrm{Out}(L)$ was $Z_2 \oplus Z_2$. $\mathrm{Out}(\pi)$ was Z_2, which once again must be the group of isotopy classes of homeomorphisms as well as the group of homotopy classes of self homotopy equivalences of Y.

7.7. We may realize H of 6.18 as the fundamental group of the orientable closed aspherical 4-manifold

$$X(\Gamma) = R \times_Z T^3.$$

The group Z generated by Γ acts upon T^3 by $\Gamma(z_1,z_2,z_3) = (z_2,z_1z_2,z_3^{-1})$. Any finite group which acts effectively on $X(\Gamma)$ must be a subgroup of $Z_2 \oplus Z_2 \oplus Z_2$.

7.8. Let $\Phi(n_1,n_2,\ldots,n_\ell) \in GL(2\ell,Z)$. There is defined a homeomorphism with fixed point on $T^{2\ell}$ by

$$\Phi(n_1,n_2,\ldots,n_\ell)(z_{11},z_{12},\ldots,z_{\ell 1},z_{\ell 2}) = (z_{11}^{n_1-1}z_{12},z_{11}^{n_1}z_{12},\ldots,z_{\ell 1}^{n_\ell-1}z_{\ell 2},z_{\ell 1}^{n_\ell}z_{\ell 2}).$$

Define

$$M^{2\ell+1}(n_1,n_2,\ldots,n_\ell) = X(\Phi(n_1,\ldots,n_\ell)) = R \times_Z T^{2\ell}.$$

We have seen from 6.20, that $\pi_1(M^{2\ell+1})$ is centerless and that

$$\text{Out}\ \pi_1(M^{2\ell+1}) = Z^{\ell-1} \oplus D_{n_1} \oplus \ldots \oplus D_{n_\ell}.$$

<u>Hence, if</u> $(G,M^{2\ell+1})$ <u>is a finite group acting effectively, then</u> $G \subset D_{n_1} \oplus \ldots \oplus D_{n_\ell}$. We also may realize the entire sum of dihedral groups $D_{n_1} \oplus \ldots \oplus D_{n_\ell}$ as an action on $M^{2\ell+1}$ by the description given in 7.3. The group D_{n_i} acts on the $2i$, and $(2i+1)$-st coordinates and trivially on the other coordinates as follows: $(\lambda^p,\delta)_i(r,z_{11},z_{12},\ldots,z_{i1},z_{i2},\ldots z_{\ell 1},z_{\ell 2}) = (r,z_{11},z_{12},\ldots,\lambda^p z_{i1}^\delta,$ $\lambda^{2p}z_{i2}^\delta,\ldots z_{\ell 1},z_{\ell 2})$, where λ is a generator of Z_{n_i} and $\delta \in Z_2$ written multiplicatively. One can check that this action is compatible with the identifications

$$(r,z_{11},\ldots,z_{i1},z_{i2},\ldots,z_{\ell 2}) \sim (r-1,z_{11}^{n_1-1}\ z_{12},\ldots,\ z_{i1}^{n_i-1}\ z_{i2},$$
$$z_{i1}^{n_i}z_2,\ldots,z_{\ell_1}^{n_\ell}z_{\ell 2}).$$

7.9. Let $\Phi(n_1,n_2,\ldots,n_\ell,a_j) \in GL(2\ell+1,Z)$ or $GL(2(\ell+1),Z)$, where $a_0 = (-1)$, $a_1 = \begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}$, $a_2 = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$ or $a_3 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. Form, $M(n_1,n_2,\ldots,n_\ell,a_j) = X(\Phi(n_1,\ldots,n_\ell,a_j)) = R \times_Z T^{2\ell+(1 \text{ or } 2)}$ similar to the previous case. This closed aspherical manifold of dimension $2(\ell+1)$ or $2\ell+1$ has centerless fundamental group and outer automorphism group as in 6.21 or 6.22.

Let us call the action of $D = D_{n_1} \oplus \ldots \oplus D_{n_\ell}$ on $M^{2\ell+1}$ described in 7.8, the "standard action" of D on M. Because $\pi_1(M) = L$ is centerless, A.4 and A.10 of the Appendix and knowledge of this action will determine the cohomology of the fixed point sets for any action (G,M) of a finite group G on M. We shall obtain generaliza-

tions of 7.2 and 7.4.

7.10. Theorem: *The effective action (G,M) has a non-empty fixed point set only if G is a subgroup of $(Z_2)^{\ell}$. In particular, if G has odd order, then G acts freely.*

Let $V = V^{2\ell+1}(n_1,n_2,\ldots,n_\ell) = M^{2\ell+1}(n_1,\ldots,n_\ell)$ minus the interior of a tame $(2\ell+1)$-ball.

7.11. Corollary: *If (G,V) denotes an effective action of a finite group then G is isomorphic to a subgroup of $(Z_2)^{\ell}$ and has non-empty fixed point set.*

Fairly simple proofs can be given if we assume that $\{n_1,n_2,\ldots,n_\ell\}$ are all odd. We first treat this case. Let (G,M) be an effective action. By the geometric realization procedure of §2 we may construct an abstract kernel $\Psi : G \to \mathrm{Out}(\pi_1(M,x)) = \mathrm{Out}(L)$. The homomorphism Ψ is an embedding of G into D. Let $r_i : D = D_{n_1} \oplus \ldots \oplus D_{n_\ell} \to D_{n_i}$ be the natural projection. For each $g \in G$, we may represent $r_i \circ \Psi(g)$ by $(\lambda^k,\delta)_i$, where λ is a generator of Z_{n_1}, k an integer $0 \leq k < n_i$, and $\delta \in Z_2$. Consider $r_i \circ \Psi(g^2) = (\lambda^k,\delta)_i\cdot(\lambda^k,\delta)_i = (\lambda^k\lambda^{\delta k},\delta^2)_i = (\lambda^{k(1+\delta)},1)_i$. Thus, g^2 is either identically $1 \in G$ or else $g^2 \in H_0(Z;Z^{\ell})$, and so the group generated by g^2 will act freely by §5. Consequently, the elements of G which fix anything must have order 2. We have not used the fact that the n_i are odd, yet.

Suppose H is a 2-group with non-empty fixed point set. Let $g,h \in H$ and $r_i \circ \Psi(g) = (\lambda^k,-1)_i$ and $r_i \circ \Psi(h) = (\lambda^{k'},-1)_i$, for some i. Then $r_i \circ \Psi(gh) = (\lambda^{k-k'},1)_i$. Since $(gh)^2 = 1$, $r_i \circ \Psi(gh)^2 = (\lambda^{2(k-k')},1)_i = (1,1)_i$. But this means that $2(k-k') = 0$ if n_i is odd. Thus, if $r_i \circ \Psi(g) = (\lambda^k,-1)_i$, λ^k is independent of g. Furthermore, it is impossible that $r_i \circ \Psi(g)$ have the form

$(\lambda^k,1)$, $k \neq 0$, for some i. For then $r_i \circ \Psi(g^2) = (\lambda^{2k},1)_i = (1,1)_i$, which implies that n_i is even. It is clear now that $H \subset (Z_2)^\ell \subset D$.

The corollary follows directly from the theorem since any effective action (G,V) extends to M by just extending the action over the interior of the deleted ball. A fixed point is introduced at the center of the added ball. Since the extension (G,M) is effective, G must be, by the theorem, isomorphic to a subgroup of $(Z_2)^\ell$. No subgroup of G may act freely on V for otherwise this would introduce an action of $(Z_2)^j$ with exactly one fixed point on the closed manifold which is impossible.

To obtain these results in the generality stated we have to apply some of the techniques and results of the Appendix. We intend to compare an arbitrary action (G,M) with the standard action (D,M). The complications arise because we must take care of the base points. The theorem is a consequence of the following considerations. Let (G,M) and $(G,M)'$ be two actions of a finite group G on an aspherical manifold with centerless fundamental group. Choose a base point $x \in M$ and construct abstract kernels Ψ, and Ψ' from the geometric realization procedure of §2.

7.12. Theorem: If $\Psi = \Psi'$, then for any p-subgroup H of G there is a one-one correspondence between the components of the fixed point sets and their cohomology groups, coefficients in Z_p, of one action with those of the other action.

The proof is fairly complicated. We shall impose the parts of the hypothesis as needed. Let (G,M) be an action of a finite group on a path connected space M. We shall assume M nice enough to admit covering space theory. Choose a base point $x \in M$ and by means of the geometric realization procedure of §2 define the homomorphism $\Psi : G \to \mathrm{Out}(\pi_1(M,x))$. Suppose that $h : (M,y) \to (M,x)$ is a homeomorphism isotopic to the identity, then $(G,M)^+$ defined by $g(m) =$

$hgh^{-1}(m)$ is equivariantly homeomorphic to (G,M). One can, by using the path from y to x given by the isotopy, check that the homomorphism Ψ^+, $\Psi^+ : G \to \mathrm{Out}\ \pi_1(M,x)$ given by the geometric realization procedure applied to $(G,M)^+$, is equal to Ψ.

Let E and E' to be induced crossed-product extensions of $\pi_1(M,x)$ defined by Ψ and Ψ'. The groups E and E' operate on the universal covering M^* of M and cover the actions (G,M) and $(G,M)'$.

7.13. Lemma: If $\pi_1(M,x)$ has trivial center then the extensions E and E' are congruent. Moreover, if M is also a finite dimensional $K(\pi,1)$ and $H \subset G$ is a p-subgroup, then the fixed point set $F = F(H,M)$ is non-empty, if and only if, $F' = F((H,M)')$ is non-empty. Finally, if M is also a manifold then there is a bijection between the components of the respective fixed point sets and an isomorphism of their cohomology groups.

Proof: Since $\pi_1(M,x)$ is assumed to have trivial center, $\Psi = \Psi'$, [6; p. 128] implies the extensions E and E' are congruent. Therefore the finite subgroups of E and E' are isomorphic.

Let y be another point in M and choose a path P_{xy} from x to y. We may use the geometric realization procedure at y to define an extension E_y of $\pi_1(M,y)$ by G on M^*. One may choose at will the particular representatives of paths from y to the various gy. However, we only need make a convenient choice and so we choose trivial paths whenever $gy = y$. Consequently, if H is a subgroup of G which leaves y fixed then E_y contains the semi-direct $\pi_1(M,y) \circ H$, (A.5). We want now to modify the extension E defined at x, E_x, slightly. The paths from x to the various gx that were used to define E_x can also be altered at will. One will get a new extension but it will be congruent to the old one since $\pi_1(M,x)$ is center-

less. One chooses the new paths so that P_{xy} induces an isomorphism of E_y onto the new E_x. The two actions E_x and E_y on M^* agree up to automorphisms of $\pi_1(M,x)$.

Suppose $H \subset G$ leaves y fixed. Then E_y contains the semi-direct product $\pi_1(M,y) \circ H$ and H leaves the base point over y fixed. Also, since $E_x \cong E_y$, E_x contains a subgroup isomorphic to H which projects to H under $E_x \to G$. Thus, some point over y is left fixed by this subgroup. We have shown that if $y \in M$ and $H \subset G_y$, then there is a subgroup isomorphic to $\pi_1(M,x) \circ H \subset E_x$, and a point in M^* over y which is left fixed by the embedding of H in E_x.

If we now assume that M is a finite dimensional $K(\pi,1)$ then Smith theory may be applied and we find that for any finite p-subgroup H of E_x, the fixed point set $F^* = F(H,M^*)$ is not empty. Also from above, if $H \subset G$ and $F = F(H,M) \neq \phi$, then $H \subset E_x$ and $F(H,M^*) \neq \phi$. We now wish to apply the results of the Appendix. Let $H \subset G$ be a p-subgroup so that $F = F(H,M) \neq \phi$ and let M also be a manifold. We may assume without loss of generality that the base point x is in F by altering M by an isotopy if necessary. Consider the other action $(G,M)'$. Since E' is naturally isomorphic to E_x, E' and E_x contain isomorphic finite subgroups. We wish to show $F' = F((H,M)')$ is not empty. Clearly there is a finite subgroup H' in E' which is isomorphic to H and projects to H in $E' \to G$. Hence, $F(H',M^*) \neq \phi$ and therefore, $F' \neq \phi$. We may also perform another isotopy of M and get an action equivalent to $(G,M)'$ but which now has $x \in F'$. We will denote it also by $(G,M)'$.

Now we have two actions (H,M) and $(H,M)'$ with $\Psi = \Psi'$ and $x \in F \cap F'$. By A.4 and A.10 of the Appendix we may conclude that the components of the fixed point sets are in one-to-one correspondence and their cohomology groups are isomorphic. This completes the proof of Lemma 7.13 and Theorem 7.12.

We can apply 7.12 in the following situation. Let (G,M) and (G_1,M) be two actions. Choose a base point x and assume that Ψ and Ψ_1 are both monomorphisms and that the images $\Psi(G)$ and $\Psi_1(G_1)$ are equal. Then, $\Psi_1^{-1} \circ \Psi : G \to G_1$ is an isomorphism. The action (G_1,M) can now be regarded as a second action of G on M, $(G,M)'$, via the isomorphism $\Psi_1^{-1} \circ \Psi$. The geometric realization procedure applied to $(G,M)'$ yields the monomorphism $\Psi' : G \to \mathrm{Out}(\pi_1(M,x))$. But observe $\Psi'(g) = \Psi_1 \circ (\Psi_1^{-1} \circ \Psi)(g) = \Psi(g)$, which guarantees the hypothesis of 7.12 holds.

We return to 7.10. Let (G,M) be any action with $\Psi : G \to \mathrm{Out}(L)$ a monomorphism into $D = D_{n_1} \oplus \cdots \oplus D_{n_\ell}$. There is also a standard action (D,M) with monomorphism $\Theta : D \to \mathrm{Out}\, L$ and Θ is an isomorphism onto the torsion subgroup of $\mathrm{Out}\, L$. We may then consider restriction of the action of D to the subgroup $\Theta^{-1} \circ \Psi(G) = G' \subset D$. The actions (G,M) and (G',M) now satisfy the preceding remark and so the fixed point set for subgroups of (G,M) are determined by knowledge of those for (D,M). From the formula in 7.8 we see that $g(y) = y$, for some $y \in M$, $g \in D$, if and only if, $g = (\lambda^k,\delta)_i = (\lambda^k,-1)_i$, or $(1,1)_i$ for each i. This will complete the proof of 7.10.

7.14. As closing remarks we mention that under fairly general assumptions, when Borel's Theorem holds, the contribution of $H_0(Z;K)$ may be always geometrically realized as a free action on $X(\Phi)$. In particular, this will work when $Y = T^k$ and $\Phi \in GL(k,Z)$ has infinite order and $\det(I-\Phi) \neq 0$. We shall discuss this result elsewhere.

The manifolds treated in this section are all manifolds whose fundamental groups are of type P, cf. C.T.C. Wall [8]. In fact, every group of type P and rank n has the form $L = \pi \circ Z$ defined by an automorphism (Φ,π) on a group π of type P and rank $(n-1)$. (A

P-group of rank 0 is the trivial group.) One of the main features of these manifolds is that they are determined by their fundamental groups and every homotopy equivalence may be geometrically realized by a homeomorphism in dimensions different from 4.

8. MANIFOLDS ON WHICH EVERY ACTION OF A FINITE GROUP IS TRIVIAL

We shall construct a family of distinct closed aspherical 4-manifolds on which every non-trivial periodic map is fixed point free. If a point is deleted from any of these manifolds no non-trivial action is possible. For, otherwise, one could extend the action to the closed manifold and the extended action would have a fixed point. The closed manifolds have the desired property because the automorphism groups of their fundamental groups have no elements of finite order other than the identity.

First, we discuss certain central extensions of Z by $Z \oplus Z$ which yield the fundamental groups of principal bundles over the torus. The closed 4-manifolds are constructed by taking particular fiberings over the circle with fibers these 3-manifolds.

Let $a \in H^2(Z \oplus Z;Z)$ be a generator of this cohomology group. We claim that the extension cocycle $f : (Z \oplus Z) \times (Z \oplus Z) \to Z$ is given by $f(p,q;r,s) = -qr$. One easily checks that f is a cocycle. To see that this extension cocycle yields the generator recall that a may be presented by $\{x,y,h : xyx^{-1}y^{-1}h^{-1}, [x,h], [y,h]\}$. The group law for the extension determined by f can be described by

$$(m;p,q)(n;r,s) = (m+n-qr;p+r,q+s).$$

Put $x = (0;1,0)$, $y = (0;0,1)$, $h = (1;0,0)$. A calculation using the cocycle f shows that $xyx^{-1}y^{-1} = h$.

Actually we shall be concerned with $2a$. An extension cocycle for $2a$ may be given as follows. Identify $(Z \oplus Z) \times (Z \oplus Z)$ with the additive group of all 2×2 integral matrices, A, by

$(p,q;r,s) \to \begin{pmatrix} p & r \\ q & s \end{pmatrix}$. The extension cocycle is $\det : A \to Z$. To see that this is a cocycle and does represent $2a$ we define $g : Z \oplus Z \to Z$ by $g(r,s) = rs$ then

$$\begin{vmatrix} p & r \\ q & s \end{vmatrix} + g(p,q) - g(p+r,q+s) + g(r,s)$$

$$= ps - qr + pq - pq - ps - rq - rs + rs$$

$$= -2qr.$$

For every integer $k > 0$ we shall consider the group extension

$$0 \to Z \to \pi(k) \to Z\oplus Z \to 1$$

corresponding to $2ka \in H^2(Z \oplus Z;Z)$. Explicitly $\pi(k)$ is the set of triples $(m;p,q)$ with

$$(m;p,q)(n;r,s) = (m+n+k(ps-rq);p+r,q+s).$$

Obviously the image of $Z \to \pi(k)$ is the center of $\pi(k)$ and hence is a characteristic subgroup.

Define an automorphism $\Phi \to \Psi$ on $GL(2,Z)$ by setting $\det(\Phi)\Psi$ equal to the inverse transpose of Φ.

8.1. <u>Lemma</u>: <u>If</u> $GL(2,Z)$ <u>acts from the left on</u> $Z \oplus Z$ <u>via the automorphism</u> $\Phi \to \Psi$ <u>then</u>

$$\mathrm{Aut}(\pi(k)) \cong (Z \oplus Z) \circ GL(2,Z).$$

<u>Proof</u>. Since $\mathrm{im}(Z \to \pi(k))$ is a characteristic subgroup we may consider an automorphism $\sigma : \pi(k) \to \pi(k)$ to be defined by $\sigma(n,\alpha) = \sigma((n,0) + (0,\alpha)) = \sigma(n,0) + \sigma(0,\alpha) = (g(n),0) + (\varphi(\alpha),\Phi(\alpha)) = (g(n) + \varphi(\alpha),\Phi(\alpha))$. Here, $n \in Z$, $\alpha \in Z \oplus Z$, $g \in GL(1,Z)$, $\Phi \in GL(2,Z)$, and φ is a unique function $\varphi : Z \oplus Z \to Z$. We wish first to show that φ is a homomorphism. Consider, $\sigma((0,\alpha)+(0,\beta)) = \sigma(0,\alpha) + \sigma(0,\beta) = (\varphi(\alpha),\Phi(\alpha)) + (\varphi(\beta),\Phi(\beta)) = (\varphi(\alpha) + \varphi(\beta) + f(\Phi(\alpha),\Phi(\beta)), \Phi(\alpha) + \Phi(\beta))$.

Also, $\sigma((0,\alpha) + (0,\beta)) = \sigma(f(\alpha,\beta),\alpha+\beta) = (gf(\alpha,\beta) + \varphi(\alpha+\beta),\ \Phi(\alpha+\beta))$. Hence, we obtain an identity

$$g(f(\alpha,\beta)) - f(\Phi(\alpha),\Phi(\beta)) = \varphi(\alpha) - \varphi(\alpha+\beta) + \varphi(\beta)$$

from which we shall now conclude that $\varphi \in \mathrm{Hom}(Z \oplus Z,Z)$. Since,

$$f(\alpha,\beta) = f(a_1,a_2;b_1,b_2) =$$

$k \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, $f(\Phi(\alpha),\Phi(\beta)) = \det \Phi\ f(\alpha,\beta)$. Also, $g(f(\alpha,\beta)) = (\det g) f(\alpha,\beta)$. But, if α and β are interchanged the left-hand side of the identity changes sign while the right-hand side remains unchanged.

Now, given any pair $(\varphi,\Phi) \in \mathrm{Hom}(Z \oplus Z,Z) \times GL(2,Z)$ we may define a corresponding $\sigma \in \mathrm{Aut}(\pi(k))$ by $(n,\alpha) \to ((\det \Phi)n + \varphi(\alpha),\ \Phi(\alpha))$. Thus we have an exact sequence

$$0 \to \mathrm{Hom}(Z \oplus Z,Z) \to \mathrm{Aut}(\pi(k)) \to GL(2,Z) \to 1.$$

We claim this sequence splits by defining for each $\Phi \in GL(2,Z)$ an element $\Phi_\#$ of $\mathrm{Aut}(\pi(k))$ by

$$\Phi_\#(n,\alpha) = ((\det \Phi)n,\Phi(\alpha)).$$

Suppose now that $\varphi : Z \oplus Z \to Z$ is a homomorphism. The corresponding automorphism of $\pi(k)$ is

$$\varphi_\#(n,\alpha) = (n+\varphi(\alpha),\alpha).$$

Now,

$$\Phi_\# \circ \varphi_\# \circ \Phi_\#^{-1}(n,\alpha) = (n+(\det \Phi)\varphi(\Phi^{-1}(\alpha)),\alpha).$$

This is the left action of $GL(2,Z)$ on $\mathrm{Hom}(Z \oplus Z,Z)$. To translate this into an action on $Z \oplus Z$ we replace $(\det \Phi)\varphi \circ \Phi^{-1}$ by $\det \Phi$ multiplied by the transpose of Φ^{-1}, that is, by Ψ.

We have shown $\mathrm{Aut}(\pi(k))$ is independent of k, however, $\mathrm{Out}(\pi(k))$ depends very much on k. Since $Z \oplus Z$ is abelian $\mathrm{Inn}(\pi(k)) \subset \mathrm{Hom}(Z \oplus Z, Z)$. Observe $f(-\alpha,\alpha) = k \det\begin{pmatrix}-p & p\\ -q & q\end{pmatrix} = 0$, thus $(0,\alpha)^{-1} = (0,-\alpha)$ and $(0,-\alpha)(m,\beta)(0,\alpha) = (m + f(\beta,\alpha) + f(-\alpha,\beta+\alpha),\beta)$, and $f(\beta,\alpha) + f(-\alpha,\beta+\alpha) = k\left(\begin{vmatrix}r & p\\ s & q\end{vmatrix} + \begin{vmatrix}-p & p+r\\ -q & q+s\end{vmatrix}\right) = k(rq-ps-qp-sp+qp+qr) = 2k(qr-sp)$. With α fixed this is the homomorphism of $Z \oplus Z$ into Z generated by conjugating with $(0,\alpha)$. Obviously, $\mathrm{Inn}(\pi(k))$ is the subgroup $2kZ \oplus 2kZ$ in $Z \oplus Z$. Consequently, we have

8.2. Lemma: For every integer $k > 0$, $\mathrm{Out}(\pi(k))$ is isomorphic to the semi-direct product

$$(Z_{2k} \oplus Z_{2k}) \circ GL(2,Z).$$

The action of $GL(2,Z)$ is induced by the natural homorphism of $GL(2,Z)$ into $\mathrm{Aut}(Z_{2k} \oplus Z_{2k})$.

We now wish to examine the centralizer of some elements in $\mathrm{Out}(\pi(k))$. Recall the matrix $\Phi = \begin{pmatrix}2k-1 & 1\\ 2k & 1\end{pmatrix}$ for which $C(\Phi) = N(\Phi)$ was the subgroup generated by Φ and $-I$. Modulo $2k$ this matrix is $\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}$. The inverse transpose is $\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ and so $\Psi = \begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$. Note that $I-\Psi = \begin{pmatrix}0 & 0\\ -1 & 2\end{pmatrix}$. Hence the cokernel of $I-\Psi$ is Z_{2k}. Let $\alpha \in Z_{2k} \oplus Z_{2k}$ represent a generator of this cokernel. We take $(\alpha,\Phi) \in \mathrm{Out}(\pi(k))$.

8.3. Lemma: $N(\alpha,\Phi) = C(\alpha,\Phi)$ and if $k > 1$ there is no element $\beta \in Z_{2k} \oplus Z_{2k}$ for which $(\beta,-I)$ lies in $C(\alpha,\Phi)$.

Proof: The first assertion follows immediately since $N(\Phi) = C(\Phi)$. Suppose such a β exists. Then

$$(\beta,-I)(\alpha,\Phi) = (\beta-\alpha,-\Phi) = (\alpha,\Phi)(\beta,-I) = (\alpha+\Psi(\beta),-\Phi).$$

But then $(I-\Psi)(\beta) = 2\alpha$ and we assumed α represented a generator of cokernel of $I-\Psi$ with $k > 1$.

8.4. Lemma: There is an isomorphism, if $k > 1$,

$$C(\alpha,\Phi)/((\alpha,\Phi)) \cong Z_{2k}.$$

Proof: We first observe that for no n does $C(\alpha,\Phi)$ contain an element of the form $(\beta,-\Phi^n)$, for if it did we could multiply by $(\alpha,\Phi)^{-n}$ to show there is a $(\gamma,-I)$ in $C(\alpha,\Phi)$. Thus we have an epimorphism

$$C(\alpha,\Phi) \to (\Phi) \subset GL(2,Z)$$

for which the kernel consists of all elements in the centralizer with the form (β,I). However, $(\beta,I) \in C(\alpha,\Phi)$ if and only if $\Psi(\beta) = \beta$. Recall that $I-\Psi = \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix}$ and so $Ker(I-\Psi) \cong Z_{2k}$. We have $0 \to Ker(I-\Psi) \to C(\alpha,\Phi) \to (\Phi) \to 0$ and $((\alpha,\Phi))$ is carried isomorphically onto (Φ). Thus

$$C(\alpha,\Phi)/((\alpha,\Phi)) \cong Ker(I-\Psi) \cong Z_{2k}.$$

Let $\lambda \in Aut(\pi(k))$ represent (α,Φ) in $Out(\pi(k))$. We form the semidirect product $L(k) = \pi(k) \circ Z$ with respect to λ. First note the action of λ on $im\ Z \to \pi(k)$ which is the center of $\pi(k)$. Since $\det \Phi = -1$ this action is $(m,0) \to (-m,0)$. Certainly (α,Φ) has infinite order in $Out(\pi(k))$ and therefore $L(k)$ has a trivial center. We must also show $I-\lambda_* : H_1(\pi(k);Q) \cong H_1(\pi(k);Q)$. But $\pi(k) \to Z \oplus Z$ induces an isomorphism

$$H_1(\pi(k);Q) \cong H_1(Z\oplus Z;Q)$$

sending λ_* into Φ_* and $\det(I-\Phi_*) = \det(I-\Phi) \neq 0$.

Thus we have the short exact sequence, if $k > 1$,

$$0 \to H_0(Z;Z) \to \mathrm{Out}(L(k)) \to C(\alpha,\Phi)/((\alpha,\Phi) \to 1$$

<u>or</u>

$$0 \to Z_2 \to \mathrm{Out}(L(k)) \to Z_{2k} \to 0.$$

If we can show $\mathrm{Aut}(L(k))$ is torsionless we are done. There is a short sequence

$$0 \to Z \to \mathrm{Aut}(L(k)) \to \Lambda \to 1$$

where Z refers to the center of $\pi(k)$ and $\Lambda \subset \mathrm{Aut}(\pi(k)) = (Z\oplus Z) \circ GL(2,Z)$ is the subgroup of elements which in $\mathrm{Out}(\pi(k))$ commute with (α,Φ). (Recall that $\mathrm{Aut}(L(k))$ are pairs (c,δ) with $c \in \mathrm{Aut}(\pi(k)), \delta \in \pi(k)$ satisfying $\mu(\delta) \circ \lambda \circ c = c \circ \lambda$. Furthermore $(c_1,\delta_1)(c_2,\delta_2) = (c_1 \circ c_2, c_1(\delta_2)\delta_1)$. Thus for a fixed c, δ is uniquely determined up to any element of the center of $\pi(k)$.) Clearly by 8.2 $\mathrm{Inn}(\pi(k)) \subset \Lambda$. There are no elements of finite order in Λ, for if there were, we would immediately conclude from the short exact sequence used in the proof of Lemma 8.4:

$$0 \to \mathrm{Ker}(I-\Psi) \to C(\alpha,\Phi) \to (\Phi) \to 0,$$

that these elements belong to $\Lambda \cap (Z \oplus Z) \subset (Z \oplus Z) \circ GL(2,Z)$. This is impossible. <u>Thus</u>, $\mathrm{Aut}(L(k))$ <u>has no elements of finite order</u>.

Now we ask if there is a closed aspherical manifold whose fundamental group is $L(k)$. This is not difficult to find. Using the fact that $Z \oplus Z$ is torsionless we construct a principal action (S^1,M) on a closed aspherical 3-manifold, with fundamental group $\pi(k)$, associated to $2ka \in H^2(Z \oplus Z,Z)$ in the manner of [5; §8]. Now any outer automorphism Ψ may be geometrically realized by a homeomorphism. To see this choose an automorphism of $\pi(k)$. This induces an automorphism Γ of $Z \oplus Z$ so that $\Gamma_*(2ka) = g_*(2ka) = \pm 2ka$. One can

then find a homeomorphism of R^2 compatible with this automorphism and hence a homeomorphism $H : M \to M$ so that $H(tm) = g_*(t)\cdot H(m)$ where $H_* : \pi(k) \to \pi(k)$ is Ψ in $\mathrm{Out}(\pi(k))$. This is essentially the same as in [5; §8] except that we allow automorphisms of S^1 in our equivariant homeomorphism. Clearly, H can be altered by an isotopy to have a fixed point. Thus there exists a homeomorphism H of M with fixed point so that the induced outer automorphism of the fundamental group is (α,Φ). (An alternative procedure is to appeal to a theorem of F. Waldhausen which says that every outer automorphism of sufficiently nice 3-manifolds is geometrically realizable as a homeomorphism.)

The required closed aspherical 4-manifold is $B(k) = (R^1 \times_Z M)$, where H is used to define $(Z, R^1 \times M)$.

Summarizing we have the

8.5. Theorem: For each $k > 1$, there exists a closed aspherical 4-manifold $B(k)$ with centerless fundamental group and

$$0 \to Z_2 \to \mathrm{Out}(\pi_1(B(k))) \to Z_{2k} \to 0.$$

Furthermore, $\mathrm{Aut}(\pi_1(B(k)))$ is torsion free, and hence every effective action of a finite group is necessarily free.

8.6. Corollary: $U(k) = B(k) - pt$ is a 4-manifold with the property that every action of a finite group is trivial.

We remark that $B(k)$ has some interesting properties:

(i) Every non-zero power of a non-trivial homotopy equivalence is never homotopic to the identity while keeping a base point fixed. However, there always exists a power dividing $4k$ of this homotopy equivalence which is homotopic to the identity without keeping a base point fixed.

(ii) There is always a free involution on $B(k)$.

(iii) $V(k) = B(k)$ - (interior of a tame 4-ball) is a compact manifold with 3-sphere boundary on which every finite group operates trivially.

(iv) Let W^4 be a contractible manifold with boundary so that $\pi_1(\partial W^4) \neq 1$. Let D^4 be a tame 4-ball in the interior of W^4. Attach a cone over ∂W to W^4 - interior D^4 along ∂W. Call this X and attach it to $V(k)$ along the 3-sphere boundary. There is a map from $V(k) \cup X$ to $B^4(k)$ which collapses X to a point and sends the interior of $V(k)$ isomorphically onto $B^4(k)$-pt $= U(k)$. This map is a simple homotopy equivalence of course. $V(k) \cup X$ is a closed triangulated integral homology 4-manifold which fails to be locally Euclidean at exactly one point. Obviously, since $V(k) \cup X$ has the same homotopy type as $B(k)$, <u>every finite group action must be trivial</u>.

APPENDIX: THE ACTION OF A P-GROUP ON ASPHERICAL MANIFOLDS AND MANIFOLDS COVERED BY SPHERES

In this section we extend the results of §3 of [5]. We adopt the notation of this paper, however, rather than the right actions of [5]. Of course, either approach yields the same results. We utilize the Smith theorems for finite p-groups acting on acyclic spaces and cohomology spheres to obtain Smith type theorems for $K(\pi,1)$'s and manifolds covered by spheres. We determine the cohomology of the fixed point set.

We consider first a pair (G,π) wherein G is a finite group acting as a group of automorphisms on a group π. For each $g \in G$ we denote by g_* the corresponding automorphism of π. We form the semi-direct product $\pi \circ G$ by introducing in $\pi \times G$ the product $(\alpha,g) \cdot (\beta,h) = (\alpha g_*(\beta),gh)$.

A crossed homomorphism $\varphi : G \to \pi$ is a function satisfying the identity $\varphi(gh) = \varphi(g)g_*\varphi(h)$ for all g,h in G. To each crossed homomorphism we may associate the graph $G_\varphi \subset \pi \circ G$ which is the image of the isomorphism $g \to (\varphi(g),g)$.

A.1. <u>Lemma</u>: If $H \subset \pi \circ G$ with $H \cap \pi = (e,1)$ then there is a subgroup $K \subset G$ and a crossed homomorphism $\varphi : K \to \pi$ whose graph is H. Moreover, if G is finite and π is torsionless then the collection of all such H is the set of all finite subgroups of $\pi \circ G$.

<u>Proof</u>: Any element in H has the form (α,g). We wish to show α depends uniquely on g. Suppose $(\beta,g) \in H$ also. Let $g_*(\gamma) = \beta^{-1}$. Then $(\beta,g)(\gamma,g^{-1}) = (e,1)$ so that $(\gamma,g^{-1}) \in H$ also. Now, however, $(\alpha,g)(\gamma,g^{-1}) = (\alpha\beta^{-1},1) \in H \cap \pi$. For the first part

then $\alpha = \beta$; for the second part we use the fact that π is torsionless; hence $\alpha = \beta$. Take $K \subset G$ to be the image of H under the natural homomorphism $\pi \circ G \to G$. Then to each $g \in K$ there uniquely corresponds an element $\varphi(g) \in \pi$ such that $(\varphi(g),g) \in H$. If $h \in K$ also $(\varphi(g),g) \cdot (\varphi(h),h) = (\varphi(g)g_*(h),gh) = (\varphi(gh),gh)$. Hence φ is a crossed homomorphism with graph H.

On the set of crossed homomorphisms of G into π we introduce an equivalence relation. We say $\varphi_0 \sim \varphi_1$ if and only if there is an $\alpha \in \pi$ such that

$$\varphi_1(g) = \alpha\varphi_0(g)g_*(\alpha^{-1})$$

for all $g \in G$.

A.2. <u>Lemma</u>: A pair of crossed homomorphisms of G to π are equivalent if and only if there is an $\alpha \in \pi$ with $G_{\varphi_1} = (\alpha,1)G_{\varphi_0}(\alpha^{-1},1)$.

<u>Proof</u>: Simply observe that

$$(\alpha,1)(\varphi_0(g),g)(\alpha^{-1},1) = (\alpha\varphi_0(g)g_*(\alpha^{-1}),g).$$

In case the group G is abelian, however, a stronger statement can be made.

A.3. <u>Lemma</u>: If G is abelian then two crossed homomorphisms of G into π are equivalent if and only if the graphs are conjugate as subgroups of $\pi \circ G$.

<u>Proof</u>: Suppose G_{φ_0} is conjugate to G_{φ_1}. Let (α,h) be the conjugating element. Put $h_*(\gamma) = \alpha^{-1}$ so that $(\alpha,h)(\gamma,h^{-1}) = (e,1)$. Then $(\alpha,h)G_{\varphi_0}(\gamma,h^{-1}) = G_{\varphi_1}$. But

$$(\alpha,h)(\varphi_0(g),g)(\gamma,h^{-1}) = (\alpha h_*(\varphi_0(g))h_*(g_*(\gamma))hgh^{-1}).$$

Using commutativity we see that since $h_*(\gamma) = \alpha^{-1}$ this last becomes $(\alpha h_*(\varphi_0(g))g_*(\alpha^{-1}),g)$. We have then arrived at the formula

$$\varphi_1(g) = \alpha h_*(\varphi_0(g))g_*(\alpha^{-1}).$$

Now $\varphi_0(g)g_*\varphi_0(h) = \varphi_0(gh) = \varphi_0(hg) = \varphi_0(h)h_*\varphi_0(g)$, or equivalently $\varphi_0(h^{-1})\varphi_0(g)g_*\varphi_0(h) = h_*\varphi_0(g)$. Put $\beta = \varphi_0(h^{-1})$, then

$$\varphi_1(g) = \alpha\beta\varphi_0(g)g_*(\beta^{-1}\alpha^{-1})$$

as required. The converse follows from Lemma 2.

In general we denote by $H^1(G;\pi)$ the set of equivalence classes of crossed homomorphisms of G into π. If π is abelian this is the usual cohomology group of course. To each crossed homomorphism $\varphi : G \to \pi$ we associate the subgroup $\Gamma_\varphi \subset \pi$ consisting of all $\beta \in \pi$ for which $\varphi(g) \equiv \beta\varphi(g)g_*(\beta^{-1})$. We note that if $\varphi_1(g) = \alpha\varphi_0(g)g_*(\alpha^{-1})$ then $\Gamma_{\varphi_1} = \alpha\Gamma_{\varphi_0}\alpha^{-1}$. Thus to each element of $H^1(G;\pi)$ we have assigned a subgroup $\Gamma_\varphi \subset \pi$ which is unique up to conjugacy. In particular, to $0 \in H^1(G;\pi)$, the equivalence class of the principal crossed homomorphisms, is assigned the subgroup of all $\beta \in \pi$ with $g_*(\beta) \equiv \beta$. Another way of saying this is that π operates on the left on the set of crossed homomorphisms by $(\alpha * \varphi)(g) = \alpha\varphi(g)g_*(\alpha^{-1})$, for $\alpha \in \pi$, $\varphi \in \mathrm{Hom}(G,\pi)$. The crossed homomorphisms on the orbit through φ are the set of crossed homomorphisms equivalent to φ. The set of orbits is the collection $H^1(G,\pi)$ and the stabilizer of φ is the subgroup Γ_φ.

We have been concerned with a fixed homomorphism $\theta : G \to \mathrm{Aut}\ \pi$ which was denoted by $\theta(g) = g_*$. However, if the center of π is trivial then given $\theta_1,\theta_2 : G \to \mathrm{Aut}\ \pi$ which agree up to inner automorphisms, that is if $\eta : \mathrm{Aut}\ \pi \to \mathrm{Aut}\ \pi / \mathrm{Inner\ Aut}\ \pi = \mathrm{Out}\ \pi$

is the natural map and $\eta\theta = \eta\theta_2$, then there is a natural equivalence $H^1_{\theta_1}(G;\pi) \approx H^1_{\theta_2}(G;\pi)$.

Specifically, we have an abstract kernel $\psi : G \to Out(\pi)$ realized by θ_1 and θ_2 and the corresponding semi-direct products $\pi \circ_{\theta_1} G$ and $\pi \circ_{\theta_2} G$. Since the center is trivial these extensions must be congruent. In particular, for each $g \in G$, there exists a unique $\mathfrak{F}(g) \in \pi$ so that

$$\theta_2(g)(\alpha) = \mathfrak{F}(g) \cdot \theta_1(g)(\alpha) \cdot (\mathfrak{F}(g))^{-1}.$$

It can be seen that

$$\mathfrak{F}(gh) = \mathfrak{F}(g) \cdot \theta_1(g)(\mathfrak{F}(h)),$$

or that $\mathfrak{F} : G \to \pi$ is a crossed homomorphism. We define an isomorphism $\tilde{\Phi} : \pi \circ_{\theta_2} G \to \pi \circ_{\theta_1} G$ by

$$\tilde{\Phi} : (\alpha,g) \to (\alpha \cdot \mathfrak{F}(g),g).$$

For each crossed θ_2-homomorphism $\varphi_2 : G \to \pi$ we may define a crossed θ_1-homomorphism $\mathfrak{F} * \varphi_2 : G \to \pi$ by

$$(\mathfrak{F} * \varphi_2)(g) = \varphi_2(g) \cdot \mathfrak{F}(g).$$

One may check that $\varphi_2 \sim \varphi_2'$ if and only if $\mathfrak{F} * \varphi_2 \sim \mathfrak{F} * \varphi_2'$. Consequently,

A.4. <u>Lemma</u>: <u>The isomorphism</u> $\tilde{\Phi} : \pi \circ_{\theta_2} G \to \pi \circ_{\theta_1} G$ <u>induces, via</u> $\mathfrak{F} *$, <u>a natural bijective correspondence between</u> $H^1_{\theta_2}(G;\pi)$ <u>and</u> $H^1_{\theta_1}(G;\pi)$. <u>Furthermore</u>, $\Gamma_{\varphi_2} = \Gamma_{\mathfrak{F} * \varphi_2}$ <u>for each crossed</u> θ_2-<u>homomorphism</u> φ_2.

We shall apply this to the following situation. Let (G,M,x) be a finite group of homeomorphisms on a path-connected space admitting a universal covering and having at least one fixed point. Taking x to

be a fixed point we obtain an action $(G,\pi_1(M,x))$ of G as a group of automorphisms on the group $\pi = \pi_1(M,x)$.

We proceed as follows. Let (M^*,b_0) be the universal covering space of M with $p : (M^*,b_0) \to (M,x)$ the covering map. We may lift the action of G to M^* with b_0 fixed and the covering map G-equivariant. We regard $\pi = \pi_1(M,x)$, the group of covering transformations, as acting on M^* from the left. There is the relation $g(\alpha b) = g_*(\alpha) \cdot (gb)$ for all $g \in G$, $\alpha \in \pi$, $b \in M^*$. We shall introduce a properly discontinuous transformation group $(\pi \circ G,M^*)$ by

$$(\alpha,g)b \equiv \alpha(gb).$$

Note that $(\alpha,g)[\beta(hb)] = \alpha g_*(\beta) \cdot (ghb) = ((\alpha,g) \cdot (\beta,h))b$. Hence the composition rule is satisfied. We see this is properly discontinuous by

A.5. Lemma: <u>At each</u> $b \in M^*$ <u>there is a canonical crossed homomorphism</u> $\varphi_b : G_{p(b)} \to \pi$ <u>whose graph is the isotropy subgroup</u> $(\pi \circ G)_b$.

If $g \in G_{p(b)}$ then $p(gb) = gp(b) = p(b)$. There is then a unique $\alpha \in \pi$ with $gb = \alpha^{-1}b$. But $\varphi_b(g) = \alpha$ so that $\varphi_b(g)(gb) = b$. Then for $h \in G_{p(b)}$ we have

$$\varphi_b(g)(g(\varphi_b(h)(hb)) = b = (\varphi_b(g)g_*\varphi_b(h)) \cdot (ghb).$$

Hence $\varphi_b(gh) = \varphi_b(g)g_*\varphi_b(h)$.

For $\alpha \in \pi$, $\varphi_{\alpha b} : G_{p(\alpha b)} \to \pi$ is the crossed homomorphism

$$\varphi_{\alpha b}(g) = \alpha\varphi_b(g)g_*(\alpha^{-1}).$$

That is, $\varphi_{\alpha b}$ is equivalent to φ_b. Observe that $(\pi \circ G)_{\alpha b} = (\alpha,1)(\pi \circ G)_b(\alpha,1)^{-1}$. If $g \in G_{p(b)}$, then $(\alpha,1)(\varphi_b(g),g)(\alpha^{-1},1) =$

$= (\alpha\varphi_b(g)g_*(\alpha^{-1}),g) = (\varphi_{\alpha b}(g),g)$.

In particular, if $p(b)$ is a fixed point of (G,M) we obtain $\varphi_b : G \to \pi$ and $G \simeq (\pi \circ G)_b = G_{\varphi_b}$.

We shall say that a crossed homomorphism $\varphi : G \to \pi$ is <u>geometrically realizable</u> if there exists $b \in M^*$, $b \neq \emptyset$ so that $(\pi \circ G)_b = G_{\varphi_b} = G_\varphi$. Observe that if φ is geometrically realized at b and $\varphi' \sim \varphi$, that is $\varphi'(g) = \alpha\varphi(g)g_*(\alpha^{-1})$ for all g, then φ' is geometrically realized at αb. For geometrically realizable crossed homomorphisms φ we shall let $F_\varphi \subset M^*$ denote the fixed point set of G_φ.

A.6. <u>Lemma</u>: $F_{\varphi_0} \cap F_{\varphi_1} \neq \emptyset$, <u>if and only if</u> $\varphi_0 = \varphi_1$ <u>and</u> $p(F_{\varphi_0}) \cap p(F_{\varphi_1}) \neq \emptyset$ <u>if and only if</u> $\varphi_0 \sim \varphi_1$.

<u>Proof</u>: Suppose $b \in F_{\varphi_0} \cap F_{\varphi_1}$, then $(\pi \circ G)_b = G_{\varphi_b}$ contains the subgroup generated by G_{φ_1} and G_{φ_0}. Obviously $G_{\varphi_0} = G_{\varphi_b} = G_{\varphi_1}$, so that $\varphi_0 = \varphi_1$. Suppose now that $p(b) \in p(F_{\varphi_0}) \cap p(F_{\varphi_1})$ and $b \in F_{\varphi_0}$. Then for some $\alpha \in \pi$, $\alpha b \in F_{\varphi_1}$; that is, $F_{\varphi_1} \cap \alpha F_{\varphi_0} \neq \emptyset$. Let $\varphi'(g) \equiv \alpha\varphi_0(g)g_*(\alpha^{-1})$ then $F_{\varphi'} = \alpha F_{\varphi_0}$ so $\varphi_1 = \varphi'$ and $\varphi_1 \sim \varphi_0$.

As a corollary we have

A.7. <u>Lemma</u>: If $F_\varphi \subset M^*$ is the fixed point set of the graph G_φ, then $\beta F_\varphi \cap F_\varphi \neq \emptyset$ if and only if $\beta \in \Gamma_\varphi$.

A.8. <u>Lemma</u>: <u>If</u> F_φ <u>is path-connected, then</u> $p(F_\varphi)$ <u>is a full path component of the fixed point set of</u> (G,M). <u>Furthermore</u>, $p(F_\varphi)$ <u>may then be identified with</u> F_φ/Γ_φ.

<u>Proof</u>: Choose a $b \in F_\varphi$ so that $\varphi = \varphi_b$. Let $c(t)$ be a path of fixed points issuing from $p(b)$ in M. There is a unique covering path $C(t)$ in M^* issuing from b. Now for any element

$(\varphi(g),g) \in G_\varphi$ the path $\varphi(g)(gC(t))$ issues from $b \in F_\varphi$, also, and by equivariance still covers $c(t)$. By uniqueness, then, $C(t) \subset F_\varphi$. Thus $p(F_\varphi)$ is a full path component in M. Since $\beta F_\varphi = F_\varphi$ for all $\beta \in \Gamma_\varphi$ we may identify the quotient F_φ/Γ_φ with the image of F_φ under the covering map $p : M^* \to M$.

More generally, if F_φ is not connected then each component maps onto a component of the fixed point set in M. If $F_\varphi(b)$ denotes a component of F_φ containing b then $\Gamma_\varphi(b) \subset \Gamma_\varphi$ denotes the invariant subgroup on $F_\varphi(b)$. The image of $F_\varphi(b)$ can be identified with $F_\varphi(b)/\Gamma_\varphi(b)$.

An immediate corollary of Lemma 8 yields

A.9. Lemma: *If each F_φ is connected then the set of path components of the fixed point set is in one-one correspondence with the subset of geometrically realizable classes of $H^1(G;\pi)$, and F_φ/Γ_φ can be identified with the image of F_φ in M.*

We intend to apply this result to two cases: G is a finite p-group and the universal covering M^* has the Z_p-cohomology of an n-sphere or π is torsionless and M^* has the Z_p-cohomology of a point. We shall also assume that M is paracompact and has finite dimension over Z_p so that we may apply the Smith theorems [2; Chap. 4]. In the first case we shall say $M^* \sim_{Z_p} S^n$ and in the latter case we shall say $M \sim_{Z_p} K(\pi,1)$.

A.10. Theorem: *If (G,M,x) is the action of a finite p-group on an $M \sim_{Z_p} K(\pi,1)$ (respectively, $M^* \sim_{Z_p} S^n$, with n and p odd), then the set of components of the fixed point set of M are in one-one*

<u>correspondence with</u> $H^1(G;\pi)$ (<u>respectively, the set of geometrically realizable classes of</u> $H^1(G;\pi)$), and $H^*(\mathrm{im}\, F_\varphi;Z_p) = H^*(F_\varphi/\Gamma_\varphi;Z_p) \cong H^*(\Gamma_\varphi;Z_p)$ (<u>respectively,</u> $H^*(\mathrm{im}(F_\varphi);Z_p) = H^*(F_\varphi^{n-2k}/\Gamma_\varphi;Z_p) \cong H^*(\Gamma_\varphi;Z_p)$ <u>for dimensions</u> $\leq n-2k$. <u>Furthermore, if</u> π <u>is finite,</u> Γ_φ <u>has periodic p-period which divides</u> $(n-2k)+1$.

Proof: By the Smith theorems each crossed homomorphism $\varphi: G \to \pi$ is geometrically realizable and $H^*(F_\varphi;Z_p) \approx H^*(pt;Z_p)$, (respectively, $H^*(F_\varphi;Z_p) \approx H^*(S^{n-2k};Z_p)$ for the φ which are geometrically realizable. The periodicity is well known.).

There is a possibility when dealing with 2-groups, in the "spherical" case, that F_φ may be exactly two disjoint acyclic components whose projections to M do not intersect. If one is willing to regard such an image as a single "component", then A.10. holds, in this modified sense, for 2-groups in the spherical case.

As a rather non-standard example of a manifold M with $M^* \sim_{Z_p} S^n$ one may take a manifold M which fibers over a manifold Y with fiber a manifold X and where either the fiber or base is aspherical and the other manifold is covered by a sphere.

For simplicity of exposition we shall restrict ourselves now to aspherical manifolds, that is manifolds which are $K(\pi,1)$'s, and manifolds covered by the sphere.

We now point out two of these results which were relevant for the actions on $K(\pi,1)$'s considered earlier in the paper.

Of course the theorem was used in the proof of 3.1. Also in the alternate version of this key step in 3.1 we referred to [5; 6.2] which can be stated as follows:

A.11. <u>Proposition</u>: <u>Let</u> (G,M^n,x) <u>be an effective action with fixed point</u> x <u>of the finite group on the closed aspherical manifold</u> M^n. <u>Then the induced homomorphism</u> $\theta : G \to \mathrm{Aut}(\pi_1(M^n,x))$ <u>must be a</u>

monomorphism.

Proof: Let K be the kernel of θ and H any p-subgroup. Consider the action (H,M,x) and F_{φ} be any component of the fixed point set. By the theorem $H^*(F_{\varphi};Z_p) \approx H^*(\Gamma_{\varphi};Z_p) \approx H^*(\pi;Z_p) = H^*(M^n;Z_p)$. If M^n is orientable then this says that dimension $F_{\varphi} = n$ and consequently as F_{φ} must be a closed cohomology submanifold it must be all of M^n. Hence, $H = e$. If M^n were not orientable one may lift the action of H to the orientable double covering.

In 7.2. we claimed that any action of Z_2 on $M(1)$ would have exactly two circles of fixed points. This now follows from Borel's theorem, A.3. and A.8.

There is a result analogous to Proposition 11 for manifolds covered by spheres. The following is a sample. Let π be a group of "type I." $\pi = \{a,b \mid a^m = 1,\ b^n = 1,\ bab^{-1} = a^r\}$, $r^n \equiv 1(m)$, $(n(r-1),m) = 1$. This is the fundamental group of a manifold of constant positive curvature. Let p be a prime divisor of m. The p-period of π is therefore $2d$ = period of π by a result of R. Swan. Let M be a closed manifold of dimension $2d - 1$, covered by the sphere and with fundamental group π (it exists). Let G be a p-group operating effectively on M and with fixed point. Then the representation $\theta : G \to \mathrm{Aut}(\pi_1(M,x))$ must be a monomorphism. If not, let K be the kernel. The group action of K can be lifted to the sphere S^{2d-1} and has as fixed point set F_{φ} a cohomology sphere over Z_p of dimension $2(d-k) - 1$. Furthermore, $\Gamma_{\varphi} = \pi$ is left invariant by θ. Thus $F_{\varphi}/\pi = F \subset M$ has dimension $2(d-k) - 1$. This also says that the p-period of π must divide $2(d-k)$ which yields a contradiction since G was assumed to be effective.

Abelian fundamental groups:

We shall examine the case $\pi_1(M,x) = \pi$ abelian. The group

$\Gamma_\varphi \subset \pi$ is independent of the crossed homomorphism $\varphi : G \to \pi$ since $\beta \in \Gamma_\varphi$ if and only if $\varphi(g) \equiv \beta - g_*(\beta) + \varphi(g)$, all $g \in G$. Thus, Γ_φ is always the subgroup of invariant elements in π. Furthermore, $H^1(G;\pi)$ is the first cohomology group.

A.12. Corollary: *If (G,M,x) is a finite p-group acting on an aspherical manifold with at least one fixed point, then each component of the fixed point set has mod p-cohomology isomorphic to $H^*(\Gamma_0;Z_p)$. If π is finitely generated (respectively, the elements of π are divisible by the order of G) then the fixed point set of G has finitely many components (respectively, exactly one component).*

Proof: If π is finitely generated then it is a free abelian group with rank $\pi \leq$ dimension M. Hence $H^1(G;\pi)$ is finite and therefore the number of components of the fixed point set, order $(H^1(G,\pi))$, is finite. If the elements of π are divisible by the order of G, then $H^1(G;\pi) = 0$. Of course, $\Gamma_0 \subset \pi$ is a torsion-free abelian group with "rank" $(\Gamma_0) \leq$ "rank" (π).

A.13. Corollary: Let (Z_p,M^n,x) *be an action with fixed point. Let p be prime, $\pi_1(M^n) \cong Z_p$ and M^n covered by the sphere. Then, there exists $k \leq p$ components of the fixed point set $C_1,C_2,\ldots,C_k$ of dimension $n_1,\ldots,n_k$. Furthermore, $H^i(C_\ell;Z_p) \approx H^i(Z_p;Z_p)$, $i < n_\ell$, and $n - k + 1 = \Sigma_{\ell=1}^{k} n_\ell$.*

This corollary has been proved by C. T. Yang, J. C. Su, J. Pak, and G. Bredon. Since $Z_p \to \mathrm{Aut}\, Z_p$ must be trivial $Z_p \circ Z_p = Z_p \times Z_p$ which acts on S^n. The number of components of the fixed point set is bounded by the order of $H^1(Z_p;Z_p) = Z_p$. The formula of Borel [2; p. 175] now allows one to obtain the expression for the dimensions.

Induced representations which become trivial in Out π:

Let (G,M,x) be an action with fixed points of a finite group on a space. We shall assume that the representation

$$\theta : G \to \mathrm{Aut}\ \pi_1(M,x)$$

has trivial image under the homomorphism

$$\eta : \mathrm{Aut}(\pi) \to \mathrm{Out}(\pi).$$

(This will always be the case if each homeomorphism $\Psi_g : (M,x) \to (M,x)$ is homotopic as a map to the identity. In particular $\eta\theta$ is trivial if (G,M,x) can be embedded in an action of a path-connected group on M.)

A.14. Proposition: *The center* $C \subset \pi_1(M,x)$ *is contained in* Γ_{φ} *for every crossed homomorphism* $\varphi : G \to \pi_1(M,x)$. *If* M *is aspherical and* G *is a p-group then each component of the fixed point of* (G,M) *has dimension greater than or equal to rank* (C).

Proof: Since $\eta\theta$ is trivial

$$\theta(g)(\alpha) = g_*(\alpha) = \beta_g\, \alpha\, \beta_g^{-1}\ ;$$

that is, β_g is unique in π/C, where C is the center. Clearly, $g_*(\alpha) = \alpha$, for all $\alpha \in C$, $g \in G$. Hence we have, for all $\alpha \in C$,

$$\varphi(g) = \alpha\varphi(g)\ g_*(\alpha^{-1}) = \varphi(g).$$

Since each component of the fixed point set of (G,M) has the form $F_{\varphi}/\Gamma_{\varphi}$ when M is aspherical and G is a p-group, we have $\dim(F_{\varphi}/\Gamma_{\varphi}) \geq \mathrm{rank}\ (C)$.

Cyclic groups:

If $G = Z_{p^k}$ is a cyclic group then there is an alternative description of $H^1(Z_{p^k};\pi)$ analogous to the π abelian case. Choose a generator $T \in Z_{p^k}$ and let T_* be the automorphism of π corresponding to T. Consider the set of elements $\delta \in \pi$ satisfying

$$\delta T_*(\delta) \ldots T_*^{p^k-1}(\delta) = e \in \pi .$$

This set is in one-to-one correspondence with crossed-homomorphisms $\varphi : Z_{p^k} \to \pi$. That is, given such a $\delta \in \pi$ and $1 \leq n \leq p^k$ set $\varphi(T^n) = \delta T_*(\delta) \ldots T_*^{n-1}(\delta)$. Correspondingly, if we are given a crossed homomorphism, φ, we take $\delta = \varphi(T)$. Naturally we say $\delta_0 \sim \delta_1$ if and only if there is an $\alpha \in \pi$ with $\delta_1 = \alpha\delta_0 T_*(\alpha^{-1})$. The set $H^1(Z_{p^k};\pi)$ is then the decomposition of the set of all $\delta \in \pi$ satisfying

$$\delta T_*(\delta) \ldots T_*^{p^{k-1}}(\delta) = e \in \pi$$

with respect to the indicated equivalence relation.

An example may be constructed as follows. Fix a prime $p > 2$, induce the curve $S \subset CP(2)$ by

$$S = \{[z_1,z_2,z_3] \,|\, z_1^p + z_2^p + z_3^p = 0\} .$$

With $\lambda = \exp(2\pi i/p)$ we define T on S by $T[z_1,z_2,z_3] = [\lambda z_1,z_2,z_3]$. The fixed point set consists, since p is odd, of the points $\{[0,-1,\lambda^i]\}_0^{p-1}$. Accordingly $H^1(Z_p;\pi_1(S))$ has exactly p-elements and for each crossed homomorphism $\varphi: Z_p \to \pi_1(S)$ the subgroup Γ_φ is trivial.

If we refer back to Lemma 3 we see

A.15. Corollary: *Let (Z_p,M,x) be a cyclic group of prime order acting as a group of homeomorphisms on an aspherical manifold with at least one fixed point. The components of the fixed point set are in*

one-to-one correspondence with the conjugacy classes of the non-trivial finite subgroups of $\pi \circ Z_p$.

From Lemma 1 every non-trivial finite subgroup of $\pi \circ Z_p$ is the graph of a crossed-homomorphism $Z_p \to \pi$. We then use Lemma 3.

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CENTRALIZERS OF ROOTLESS INTEGRAL MATRICES

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I. INTRODUCTION

In this note we present the construction of two compact connected 4-manifolds without boundary, M_1 and M_2, satisfying:

(1) M_1 is orientable, and the only periodic homeomorphisms of M_1 have period 2, and these reverse the orientation.

(2) M_2 is non-orientable, and the only periodic homeomorphisms of M_2 have period 2.

(3) M_1 and M_2 have contractible universal covering spaces and centerless fundamental groups.

The construction depends on the following result of A. Borel [2].

If M is a compact connected manifold without boundary whose universal covering space is contractible, and the center of $\pi_1 M$ is trivial, then there is associated canonically to every effective action of a finite group G on M, an embedding of G into $\mathrm{Out}(\pi)$.

Let φ be an automorphism of the 3-torus T^3, and regard φ as an element of $GL(3,Z)$. We construct M from $T^3 \times [0,1]$ by identifying $(t,0)$ with $(\varphi(t),1)$. We choose $\varphi = \begin{pmatrix} [illegible] & [illegible] & 1 \\ 0 & 1 & 1 \\ 1 & [illegible] & [illegible] \end{pmatrix}$ and will establish the following:

(1) φ and $I-\varphi$ are non-singular.

(2) φ does not have finite order.

(3) φ is not conjugate to φ^{-1}.

(4) $C(\varphi)/(\varphi)$ has torsion subgroup Z_2, where $C(\varphi)$ denotes the centralizer of φ in $GL(3,Z)$.

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By results of P. E. Conner and F. Raymond [2], (1) and (2) imply that $\mathrm{Out}(\pi_1 M) \approx N(\varphi)/(\varphi)$, where $N(\varphi)$ is the normalizer of φ in $GL(3,Z)$, and also M has a contractible universal covering space, and $\pi_1 M$ has a trivial center. By (3), $N(\varphi) = C(\varphi)$, and it follows that M satisfies the conditions for M_1 if $\det \varphi = 1$, and for M_2 if $\det \varphi = -1$.

We are indebted to Professor P. E. Conner for pointing out these facts, and for bringing the problem of determining φ to our attention. We also thank Professors A.H. Clifford and L. Fuchs for several valuable suggestions which shortened our original arguments.

2. A CLASS OF ROOTLESS MATRICES

We will show that $A_0 = \begin{pmatrix} 0&0&1 \\ 0&1&1 \\ 1&1&1 \end{pmatrix}$ is rootless in $GL(3,Z)$, i.e., $B^m = A_0$ is not satisfied with $m > 1$, $m \in \omega$. The proof can be carried through almost as simply for matrices in the class $\mathcal{C}$ defined as follows: $\mathcal{C}$ consists of those matrices in $GL(3,Z)$ with determinant -1 whose characteristic roots x_1, x_2, x_3 satisfy

$$-1 < x_1 < 0 < x_2 < |x_1| < 1 < x_3 .$$

For $A \in GL(3,Z)$ with $\det A = -1$, let the characteristic polynomial be $f(x) = x^3 + ax^2 + bx + 1$. (In the case of A_0, $a = -2$ and $b = -1$; $x_1 = -.8$, $x_2 = .54$, $x_3 = 2.25$ approximately.) We note

(I) $\mathcal{C}$ is closed under odd integral powers and odd integral roots.

This follows from the fact that the mappings $x \to x^k$ and $x \to x^{1/k}$ are order preserving for k odd. In the case of odd roots, the characteristic roots are again all real, by a consideration of the Jordan normal form.

(II) $A \in \mathcal{C} \Leftrightarrow a < b < 0$, a,b integers, and $\det A = -1$.

Proof. Suppose $A \in \mathcal{C}$, and visualize the graph of $y = f(x)$. Evidently $f(-1) = a-b < 0$, $f(1) = a+b+2 < 0$, and it follows that

$-a = x_1+x_2+x_3$ lies in the interval (x_2,x_3). Hence $f(-a) = 1-ab < 0$, so $ab > 1$. Thus a and b are both negative, and it follows that $a < b < 0$.

Conversely, if $A \in GL(3,Z)$, $\det A = -1$, and $a < b < 0$, then $f(-1) = a-b < 0$, $f(0) = 1$, and $f(1) = a+b+2 < 0$. Hence $-1 < x_1 < 0 < x_2 < 1 < x_3$. Also $x_1+x_2+x_3 = -a$ and $f(-a) = {}$ $= 1 -ab < 0$, so $x_1+x_2 < 0$ and $A \in \mathcal{C}$.

(III) If $B \in \mathcal{C}$, k odd, and $B^k = A$, then $[T_A^{1/k}]-1 \leq T_B \leq [T_A^{1/k}]$.

<u>Proof</u>. We note that $T_A = [x_3]$. Let the characteristic roots of B be y_1,y_2,y_3, with $y_i^k = x_i$; then $T_B = [y_3]$. First, $y_3^k < (1+y_3)$, so $T_A = [y_3^k] \leq (1+y_3)^k$. Hence $T_A^{1/k} \leq 1+y_3$, so $[T_A^{1/k}] \leq [1+y_3] = 1 + [y_3] = 1 + T_B$. Second, $[y_3] \leq y_3$, so $[y_3]^k \leq y_3^k$. Therefore $T_B^k = [y_3]^k \leq [y_3^k] = T_A$ so $T_B \leq T_A^{1/k}$, and $T_B \leq [T_A^{1/k}]$.

<u>Theorem 1</u>. <u>If</u> $A \in \mathcal{C}$ <u>and trace</u> $A \leq 7$ <u>then</u> A <u>has no integral roots</u>.

<u>Proof</u>. Let $A \in \mathcal{C}$, $k \in \omega$, and suppose $B \in GL(3,Z)$ satisfies $B^k = A$. Note that k must be odd, since $\det A = -1$. Thus by (I), $B \in \mathcal{C}$. Let the characteristic polynomial of B be $x^3-p_1x^2-p_2x+1$. By (II), $p_1 < p_2 < 0$, so $-p_1 \geq 2$. By (III), $T_B \leq [T_A^{1/k}]$ so if $T_A \leq 7$ we get $2 \leq -p_1 \leq [7^{1/k}] = 1$ for $k \geq 3$. Thus A has no roots.

Note

(1) If A is rootless, so is A^{-1}

(2) If A (of odd order) is rootless and has real characteristic roots, then $-A$ is rootless.

<u>Proof</u>. (1) is obvious. To see (2), suppose $B^{2k+1} = -A$ with $k > 1$. Then $(-B)^{2k+1} = A$, a contradiction. If $B^{2k} = -A$, then $\det(-A) = 1$, so $\det(A) = -1$. Thus, A has a negative characteristic root, but the characteristic roots of B^{2k} are all positive, a contradiction.

3. THE CENTRALIZER

We denote by $C(A)$ the centralizer of A in $GL(3,Z)$.

Theorem 2. *If A is a rootless matrix in $GL(3,Z)$ with distinct real irrational characteristic roots then $C(A) \approx (A) \times (-I) \times N$, where N is free abelian, and rank $N \leq 1$. Thus, the Torsion subgroup of $C(A)/(A)$ is isomorphic with Z_2 .*

Proof. Since A has distinct characteristic roots, the rational matrices which commute with A are the rational polynomials in A , i.e., $C_Q(A) = Q[A]$ [1, p. 282] . Let $p(x)$ be the characteristic polynomial of A . Then $p(x)$ is irreducible in $Q[x]$, and we have $C_Q(A) = Q[A] \approx \frac{Q[x]}{(p(x))}$ is a field. Let I denote the ring of algebraic integers of $Q[A]$, and let U be the group of units of I . Then $C(A) \subset U$, and by Dirichlet's theorem we have $U \approx Z_2 \times Z \times Z$ [4]. Thus $C(A)$ is commutative and finitely generated, with torsion subgroup $(-I)$. Let $C_+(A)$ be those elements of $C(A)$ with determinant 1 . Then $C(A) \approx C_+(A) \cdot (-I)$ (direct product). Further $C_+(A)$ is free abelian with rank ≤ 2 . If $\det A = 1$, then since A is rootless in $C_+(A)$ it follows that (A) is a pure subgroup of $C_+(A)$, and hence is a direct factor [3]. Thus $C(A) = (-I)\cdot(A)\cdot N$ (direct product). If $\det A = -1$, then $\det(-A) = 1$, and $(-A)$ satisfies the other assumptions on A . Thus $C(A) = C(-A) =$ $= (-I)\cdot(-A)\cdot N = (-I)\cdot(A)\cdot N$, and the proof is complete.

4. SUMMARY

As we have shown, the matrix $A_0 = \begin{pmatrix} 001 \\ 011 \\ 111 \end{pmatrix}$ satisfies (4) of the introduction. It is readily seen that (1),(2), and (3) also hold, thus the manifold M_2 exists as claimed. To construct M_1 we consider the matrix $-A_0$, whose determinant is $+1$. Again, the properties (1),(2), and (3) are easily verified, and (4) follows from Theorem 2 using the fact that $-A$ is rootless. Some other matrices

A such that $C(A)/(A)$ has torsion subgroup Z_2 are $A_1 = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 5 & -7 & 6 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Here $\det A_1 = 1$, and A_1 has one positive real eigenvalue, and 2 negative real eigenvalues ;

$\det A_2 = 1$, and A_1 has one positive real eigenvalue, and two complex eigenvalues. In this case we obtain $C(A_2)/(A_2) \approx Z_2$.

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MANIFOLDS WITH NO PERIODIC MAPS

by

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This paper is a sequel to Manifolds with Few Periodic Homeomorphisms [2]. We shall construct in §9 and §11 closed connected aspherical manifolds which admit no effective actions of any non-trivial finite group. The dimensions of these examples will be 7, 11, 16, 22, 29 and 37.

In §8 we shall construct, for each dimension $n+1 \geq 3$, closed aspherical manifolds N^{n+1} for which only the group of two elements, $\mathbb{Z}_2$, among the finite groups, acts effectively. The fixed point set must be 1-dimensional. For $n+1 > 3$, N^{n+1} can be chosen orientable. Consequently, N^{2m} admits no orientation preserving non-trivial periodic homeomorphisms.

Basic for our constructions is the determination of the centralizers of certain "fundamental" matrices, Φ, in $GL(n, Z)$. What is desired is that the centralizer $C(\Phi)$ of Φ be a finitely generated abelian group whose only torsion element is $-I$. Furthermore, the matrix Φ must generate a free summand of $C(\Phi)$ and so the quotient group, $C(\Phi)$ by the group generated by Φ, will be torsion free except for a $\mathbb{Z}_2$ summand generated by $-I$. Now those matrices in $\mathrm{Mat}(n, \mathbb{Q})$ which commute with Φ may be identified with a certain number field $\mathbb{Q}(\lambda)$. The group $C(\Phi)$ may be identified with a subgroup of the group of units of the ring, R, of algebraic integers of this field and Φ itself should be a fundamental unit of R. This is treated in §2. A list of such matrices in each dimension is given.

To produce $N^{n+1}(\Phi)$ we select, as in [2; §7], a homeomorphism, with fixed point, of the n-dimensional torus, T^n,

$$h: T^n \longrightarrow T^n$$

so that

$$h_*: \pi_1(T^n, *) \longrightarrow \pi_1(T^n, *)$$

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is the automorphism $\Phi \in GL(n, \mathbb{Z})$. We form the mapping torus

$$\mathbb{R}^1 \times_{(h)} T^n = N^{n+1}(\Phi) .$$

By insisting that $\det(I - \Phi) = \pm 1$, we can, by using the results of [2], show that

(i) $\pi_1(N^{n+1}(\Phi))$ has trivial center

(ii) $\operatorname{Out} \pi_1(N^{n+1}(\Phi)) \approx \mathbb{Z}_2$.

Since $N^{n+1}(\Phi)$ is a closed aspherical manifold, Borel's theorem implies that any effective finite action must be an involution.

For $n+1 = 4$, this part of our paper overlaps with the paper of R. S. Koch and G. Pall in these Proceedings.

The descriptions of the closed manifolds M with no periodic homeomorphisms is more complicated. But, just as the construction of $N^{n+1}(\Phi)$ is modelled after our [2; §7], the construction of M is a generalization of [2; §8]. We proceed as follows. Let $N = \mathbb{Z}^n$ and $V = N \otimes \mathbb{R}^1 = \mathbb{R}^n$. On the cartesian product $G = \Lambda^2(V) \times V$ we introduce a Lie group structure by

$$(\vec{\alpha}, \vec{u}) \cdot (\vec{\beta}, \vec{v}) = (\vec{\alpha} + \vec{\beta} + \vec{u} \wedge \vec{v},\ \vec{u} + \vec{v}) .$$

Obviously there is a discrete subgroup $\pi = \Lambda^2(N) \times N \subset G$ and the homogeneous space G/π is a closed aspherical $\left((n^2+n)/2\right)$-manifold with fundamental group π. The automorphism group, $\operatorname{Aut}(\pi)$, is shown to be a semi-direct product $\operatorname{Hom}(N, \Lambda^2(N)) \circ GL(n, Z)$ where the left action $\left(GL(n, Z), \operatorname{Hom}(N, \Lambda^2(N))\right)$ is given by $\Phi'(h)(v) = \Lambda^2(\Phi)\left(h(\Phi^{-1}(v))\right)$ and where $\Phi \in GL(n, Z)$ and $h: N \to \Lambda^2(N)$ is a homomorphism. The outer automorphism group, $\operatorname{Out}(\pi)$, is also described and, significantly, this has a trivial center.

An element $(h, \Phi) \in \operatorname{Aut}(\pi)$ extends to an automorphism of G under which π is invariant. Simply extend h to a real linear transformation $h: V \longrightarrow \Lambda^2(V)$, note that $GL(n, \mathbb{Z}) \subset GL(n, \mathbb{R})$ and then put

$$(\vec{\alpha}, \vec{u}) \longrightarrow \left(\Lambda^2(\Phi)(\vec{\alpha}) + h(\Phi(\vec{u})),\ \Phi(\vec{u})\right) .$$

This automorphism of G will induce a homeomorphism $\Psi: G/\pi \to G/\pi$ which leaves $e \in G/\pi$ fixed. Furthermore $\Psi_*: \pi_1(G/\pi, e) \cong \pi_1(G/\pi, e)$ is just the element of $\operatorname{Aut}(\pi)$ with which we began. We should note the analogy that elements of $\operatorname{Aut}(\pi)$ are represented by homeomorphisms of G/π just as elements of $GL(k, Z)$ are represented by homeomorphisms of a k-torus.

A closed $(n^2+n+2)/2$-manifold M is constructed by forming the cylinder $I \times (G/\pi)$ and making the identification $(0, y) \sim (1, \Psi(y))$, that is, forming the mapping torus of the map Ψ. The resulting M fibers over S^1 with fiber G/π and structure group $\mathbb{Z}$. The fundamental group $\pi_1(M)$ is the the semi-direct product $\pi \circ Z$ formed with respect to the automorphism $(h, \Phi) \in \mathrm{Aut}(\pi)$.

If Φ has infinite order in $GL(n, \mathbb{Z})$ and $\det(I - \Lambda^2(\Phi)) \cdot \det(I - \Phi) \neq 0$ then $\pi \circ Z = L$ has a trivial center and π is a characteristic subgroup. Thus the procedure in [2] can be applied to study $\mathrm{Out}(L)$. It turns out for $n = 3$ that if $\det(I - \Lambda^2(\Phi)) \cdot \det(I - \Phi) = \pm 1$ then $\mathrm{Out}(L)$ is isomorphic to a certain subgroup of the quotient $C(\Phi)/(\Phi)$. Here $C(\Phi) \subset GL(3, Z)$ is the centralizer of Φ and (Φ) is the infinite cyclic subgroup generated by Φ. The subgroup of $C(\Phi)/(\Phi)$ to which $\mathrm{Out}(L)$ is isomorphic depends on the homomorphism $h: N \to \Lambda^2(N)$. With only a few listed exceptions (see §11), it is possible to choose h so that the image of $\mathrm{Out}(L)$ misses the element of order 2 in $C(\Phi)/(\Phi)$ represented by $-I$. On the other hand, it is not uncommon to find that $C(\Phi)/(\Phi)$ is an abelian group whose only element of finite order is that represented by $-I$. In fact, in addition to the list for all dimensions in 2. 12, we give in §11 an infinite list of matrices in $SL(3, \mathbb{Z})$ with exactly this property. About half of these matrices in $SL(3, \mathbb{Z})$ can be chosen with all positive eigenvalues. This enables us to show (§10) that the corresponding M^7 can be constructed as <u>closed aspherical solvmanifolds</u>.

The cases where at most one real eigenvalue exists yields M^7 with $\mathrm{Out}(\pi_1(M^7)) = 0$. This means that <u>every self homotopy equivalence of M is freely homotopic to the identity</u>. Interestingly enough there are self homotopy equivalences on M^7 (and/or homeomorphisms) all of whose powers are never homotopic, through base point preserving homotopies, to the identity.

Good choices for $\Phi \in SL(3, \mathbb{Z})$ are

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -a-1 & a \end{pmatrix}$$

where

$$2a+3 \text{ is prime}$$

and $\quad a \equiv 1, 2, 3 \pmod 5 \quad \text{or} \quad a \equiv 1, 2, 3, 6 \pmod 7$

and $\quad |a+3| > 3(|a|+2)^{1/3}$.

If $a \geq 5$, then we get examples of closed solvmanifolds with no periodic maps.

The matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

yields an orientable closed aspherical M^7 with $\operatorname{Out} \pi_1(M^7) = 0$.

In all dimensions it really appears necessary, in our approach, to make an additional computation to show $\operatorname{Out}(L)$ is isomorphic to a subgroup of $C(\Phi)/(\Phi)$ which misses $-I$. This takes the form of showing $\det(I - \Phi')$ is odd $\neq \pm 1$, where Φ' is an automorphism, $\operatorname{Hom}(N, \Lambda^2(N)) \longrightarrow \operatorname{Hom}(N, \Lambda^2(N))$ and defined by $\Phi'(h) = \Lambda^2(\Phi) \circ h \circ \Phi^{-1}$. This condition in dimension 3 simplifies to just avoiding several values of trace (Φ). (These are the listed exceptions mentioned above.) In higher dimensions the computation of $\det(I - \Phi')$ and $\det(I - \Lambda^2(\Phi))$ are made in terms of the eigenvalues of Φ on an electronic computer. Clearly we could have investigated many more dimensions than those listed in §7.

We also would like to mention that Edward Bloomberg, in his doctoral thesis, has constructed orientable closed 4-manifolds on which no finite group can operate. These manifolds are obtained by taking the oriented connected sum of distinct closed aspherical manifolds $B(k) \# B(k') = B$ of [2; §8]. They are not aspherical and the proof that they admit no action uses some different and interesting techniques.

2. Fundamental matrices in $GL(n, \mathbb{Z})$

For the constructions to be employed in the remaining parts of the paper we shall require matrices Φ in $GL(n, \mathbb{Z})$ satisfying:

2.1 $\det(\Phi) = \pm 1$,

2.2 $\det(I - \Phi) = \pm 1$,

2.3 for no $\Psi \in GL(n, \mathbb{Z})$ is $\Psi \Phi \Psi^{-1} = \Phi^{-1}$,

2.4 the only elements of finite order in $C(\Phi)$, the centralizer of Φ in $GL(n, \mathbb{Z})$, are $\pm I$,

2.5 for no $\Psi \in GL(n, Z)$ and no $k > 1$, is $\Phi = \pm \Psi^k$.

A matrix Φ satisfying (2, 1), ..., (2. 5) will be called a <u>fundamental matrix in</u> $GL(n, \mathbb{Z})$.

If the eigenvalues of Φ are distinct, and the characteristic polynomial of Φ, $f(x)$, is irreducible, then the above conditions are implied by certain conditions on $f(x)$ which are derived below. These conditions are not all necessary conditions.

Henceforth let $f(x)$ be a monic irreducible polynomial with integral coefficients. Let its zeros, which are necessarily distinct, be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Assume that $f(x)$ is the characteristic polynomial of Φ. Clearly, (2. 1) and (2. 2) are equivalent to

2.6 $$f(0) = (-1)^n(\pm 1) .$$

2.7 $$f(1) = \pm 1 .$$

Now the characteristic polynomial of $\Psi\Phi\Psi^{-1}$ is $f(x)$, while that of Φ^{-1} is $\prod_{i=1}^{n}(x - \lambda_i^{-1}) = (-1)^n x^n f(1/x)/\det(\Phi)$. Hence, (2. 3) is implied by

2.8 $$f(x) \neq x^n f(1/x)/f(0) .$$

If

$$f(x) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \ldots + (-1)^n \sigma_n ,$$

then

$$(-1)^n x^n f(1/x) \big/ \det(\Phi) = x^n f(1/x) / f(0) = x^n - \frac{\sigma_{n-1}}{\sigma_n} x^{n-1} + \ldots + (-1)^n \frac{1}{\sigma_n} . ,$$

Thus, (2. 8) can be checked by looking at the coefficients of $f(x)$.

Since the λ_i are distinct, a result of linear algebra [3; p. 107] says that the only matrices (with rational coefficients) which commute with Φ can be written as polynomials in Φ. Since $f(\Phi) = 0$, each polynomial may be taken modulo $f(x)$, and so may be taken to have degree less than n. Further, this representation is <u>unique</u>, for if $g(\Phi) = h(\Phi)$, then $g(x) - h(x) = f_1(x)$ has degree less than n. Since $f_1(\Phi) = 0$, $f_1(x) = 0$. Hence we have a ring isomorphism:

$$Z(\Phi) = \text{centralizer } (\Phi) \text{ in Matrices } (n, \mathbb{Q}) \longrightarrow \mathbb{Q}[x]\big/(f(x)) ,$$

given by

$$\Psi = g(\Phi) \longrightarrow g(x) \text{ modulo } f(x).$$

Since $f(x)$ is irreducible, $(f(x))$ is a prime ideal, and so $\mathbb{Q}[x]/(f(x))$ is a field which is isomorphic to $\mathbb{Q}(\lambda)$, where λ is any zero of $f(x)$. Hence we have an isomorphism, α, of fields

2.9 $$\alpha: Z(\Phi) \longrightarrow \mathbb{Q}(\lambda)$$

<u>given by</u>

$$\Psi = g(\Phi) \longrightarrow \alpha(\Psi) = g(\lambda) .$$

Since this is an isomorphism, the minimal polynomial of Ψ in Matrices $(n, \mathbb{Q})$ must be the same as the minimal polynomial of $\alpha(\Psi)$ over $\mathbb{Q}$. Now if $\Psi \in C(\Phi) = Z(\Phi) \cap GL(n, \mathbb{Z})$, then $\Psi^{-1} \in GL(n, \mathbb{Z})$ and their minimal polynomials both have integer coefficients. Hence both $\alpha(\Psi)$ and $\alpha(\Psi^{-1}) = \alpha(\Psi)^{-1}$ are algebraic integers. Therefore $\alpha(\Psi)$ is a unit in the ring, R, of integers of $\mathbb{Q}(\lambda)$. <u>Hence</u> $C(\Phi)$ <u>is isomorphic with a subgroup of the group of units of</u> R. Let r_1 be the number of real zeros of $f(x)$. Then Dirichlet's unit theorem [1; p. 112] says that the group of units of R is isomorphic with $J^r \times W$, where J is an infinite cyclic group and W is the finite group of roots of one in R. Here $r = r_1 + (n - r_1)/2 - 1$. <u>Therefore</u> (2.4) <u>will hold if</u>

2.10 $\mathbb{Q}(\lambda)$ <u>contains no complex roots of one.</u>

The isomorphism in (2.9) shows that (2.5) <u>is true if</u>

<u>there is no</u> $\beta \in R$ <u>with</u> $\alpha(\Phi) = \pm\beta^k$ <u>for</u> $k > 1$.

It is convenient to use a stronger condition. If there is such a β, then it is a root of $f(\pm x^k)$. Since $\beta \in R$, its minimal polynomial over $\mathbb{Q}$ must have degree dividing n, and this minimal polynomial must divide $f(\pm x^k)$.

Hence (2.5) is implied by (2.11):

2.11 <u>For all</u> $k > 1$, $f(\pm x^k)$ <u>is irreducible.</u>

We now give examples of polynomials which satisfy conditions (2.6), (2.7), (2.8), (2.10), (2.11). For the rest of this section let $f(x)$ refer to one of the polynomials in the following list.

$\underline{n = 2}$

$x^2 - x - 1$

$\underline{n > 2}$

2.12 $\quad n \equiv 1 \pmod 2 \quad x^n - x - 1, \quad x^n - x + 1$

$n \equiv 2 \pmod 6 \quad x^n - x^3 + 1, \quad x^n + x - 1$

$n \equiv 4, 0 \pmod 6 \quad x^n - x + 1, \quad x^n + x - 1$

Note that if $n > 2$, we have both signs for $f(0)$. It is clear that conditions (2.6), (2.7), (2.8) are all satisfied for these $f(x)$.

(2.13). Ljunggren's theorem [4]: Let $g(x) = x^n + \epsilon x^m + \epsilon'$ with $d = \gcd(n, m)$, $m = m_1 d$, $n = n_1 d$, $\epsilon = \pm 1$, $\epsilon' = \pm 1$, $n \geq 2m$. Then $g(x)$ is irreducible unless $n_1 + m_1 \equiv 0 \pmod 3$ and one of the following three conditions holds:

n_1, m_1 both odd and $\epsilon = 1$;

n_1 even and $\epsilon' = 1$;

m_1 even and $\epsilon' = \epsilon$.

This theorem may be applied to show that all of the given $f(x)$ are irreducible and that condition (2.11) holds.

If $f(x)$ has a real zero, λ, then obviously $\mathbb{Q}(\lambda)$ is contained in the real numbers, so it can contain no complex roots of one. This simple observation shows that (2.10) is true if n is odd or if $f(0) < 0$.

Hence we will have shown that any matrix Φ whose characteristic polynomial is one of the $f(x)$ given above satisfies conditions (2.1), (2.2), (2.3), (2.4), (2.5) when we have shown that (2.10) is true for $n \equiv 2 \pmod 6$, $f(x) = x^n - x^3 + 1$ and $n \equiv 4, 0 \pmod 6$, $f(x) = x^n - x + 1$. These two cases will also be useful in constructing certain orientable manifolds later.

Suppose now that $\mathbb{Q}(\lambda)$ contains a primitive p-th root of one, ζ_p, for some odd prime p. Now every norm from $\mathbb{Q}(\lambda)$ must also be a norm from $\mathbb{Q}(\zeta_p)$. By a standard theorem on cyclotomic fields [1; p. 327], each such number must be congruent to either 0 or 1 modulo p. In the cases under discussion, $f(-1) = \text{norm}\left(-1 - \alpha(\Phi)\right) = 3$, so $p = 3$. In $Q(\zeta_3)$, the principal ideal $(3) = (2\zeta_3 + 1)^2$, so that (3) must be the square of an ideal of R. By a standard result in algebraic number theory [1; p. 203], this cannot happen if $f(x)$ has no

multiple factors considered as a polynomial in $\mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the field with 3 elements. In any field, this condition can be checked by showing that if $g(x) = \gcd(f(x), f'(x))$, then $g(x) = 1$. In the first case $g(x)$ must divide $f'(x)$, which is $2x^{n-1}$ in $\mathbb{F}_3[x]$. Hence $g(x)$ must be a power of x which divides $f(x)$, so $g(x) = 1$. In the second case, if $n \equiv 0 \pmod 6$, then $g(x)$ must divide $f'(x)$, which is $nx^{n-1} - 1 = -1$ in $\mathbb{F}_3[x]$, so $g(x) = 1$. In the remaining case $g(x)$ must divide $f(x) - xf'(x) = x^n - x + 1 - x(x^{n-1} - 1) = 1$. This contradiction completes the discussion of the given $f(x)$. Thus, in particular, Φ <u>is a fundamental matrix in</u> $GL(n, \mathbb{Z})$ <u>if its characteristic polynomial appears in our list</u> (2.12).

3. <u>The group</u> π

Let $N = \mathbb{Z}^n$ be a free $\mathbb{Z}$-module of rank n, and let $K = N \wedge N$ be the 2-fold exterior product. Define a group extension:

$$0 \longrightarrow K \longrightarrow \pi \longrightarrow N \longrightarrow 0 ,$$

with multiplication in π given by

$$(\alpha, u)(\beta, v) = (\alpha + \beta + u \wedge v, u + v) .$$

(3.1) <u>Lemma</u>: <u>The center of</u> π <u>is the subgroup</u> K

<u>Proof</u>: $(\alpha, u)(\beta, v)(-\alpha, -u) = (\beta + 2u \wedge v, v)$, hence (α, u) is in the center if and only if $2u \wedge v = 0$ for all $v \in N$. Hence $u = 0$, which implies that K is the center of π.

Because K is the center of π, it is a characteristic subgroup of π, and any automorphism of π restricts to an automorphism of K. The projection from π to N induces a homomorphism $\mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(N)$. For each $h \in \mathrm{Hom}(N, K)$, let $\sigma_h \in \mathrm{Aut}(\pi)$ be given by

$$\sigma_h(\alpha, u) = (\alpha + h(u), u) .$$

(3.2) <u>Lemma</u>: <u>The map</u> $h \longmapsto \sigma_h$ <u>makes the following sequence exact.</u>

$$1 \longrightarrow \mathrm{Hom}(N, K) \longrightarrow \mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(N).$$

Proof: Exactness at Hom(N, K) is obvious. Suppose $\sigma \in \operatorname{Ker}\left(\operatorname{Aut}(\pi) \longrightarrow \operatorname{Aut}(N)\right)$. Then $\sigma(\alpha, u) = \sigma(\alpha, 0)\sigma(0, u) = \left(\sigma(\alpha), 0\right)\left(h_\sigma(u), u\right)$ $= \left(\sigma(\alpha) + h_\sigma(u), u\right)$ for some $h\colon N \to K$, not necessarily a homomorphism. Since $\sigma \in \operatorname{Aut}(\pi)$, we see that

$$\sigma(\alpha + \beta + u \wedge v) + h_\sigma(u+v) = \sigma(\alpha) + \sigma(\beta) + h_\sigma(u) + h_\sigma(v) + u \wedge v .$$

consequently,

$$\sigma(u \wedge v) - u \wedge v = h_\sigma(u) + h_\sigma(v) - h_\sigma(u+v) .$$

This equation has an alternating function on $N \times N$ equal to a symmetric one, so both must be zero. Therefore the restriction of σ to K is the identity and $h_\sigma \in \operatorname{Hom}(N, K)$.

Let $\{e_i,\ 1 \le i \le n\}$ be a $\mathbb{Z}$-basis of N, so that $\{e_i \wedge e_j,\ 1 \le i < j \le n\}$ is a $\mathbb{Z}$-basis of $N \wedge N = K$. Then any $\Phi \in \operatorname{Aut}(N)$ induces an automorphism $\Phi \wedge \Phi$ of K, determined by $\Phi \wedge \Phi(e_i \wedge e_j) = \Phi(e_i) \wedge \Phi(e_j)$.

Now we can define a homomorphism $\operatorname{Aut}(N) \longrightarrow \operatorname{Aut}(\pi)$, $\Phi \longmapsto \sigma_\Phi$, where

$$\sigma_\Phi(\alpha, u) = \left(\Phi \wedge \Phi(\alpha),\ \Phi(u)\right) .$$

It is easy to see that this map splits the exact sequence of Lemma 3.2, so that the following sequence is split exact:

$$1 \longrightarrow \operatorname{Hom}(N, K) \longrightarrow \operatorname{Aut}(\pi) \underset{\longleftarrow}{\longrightarrow} \operatorname{Aut}(N) \longrightarrow 1 .$$

This makes $\operatorname{Aut}(\pi)$ naturally isomorphic with the semi-direct product $\operatorname{Hom}(N, K) \circ \operatorname{Aut}(N)$.

We need to know how Aut(N) acts as a group of automorphisms on Hom(N, K). If $\Phi \in \operatorname{Aut}(N)$, write the induced map as $\Phi'\colon \operatorname{Hom}(N, K) \longrightarrow \operatorname{Hom}(N, K)$.

(3.3) Lemma: $\Phi'(h) = \Phi \wedge \Phi \circ h \circ \Phi^{-1}$, where $\circ$ represents composition of functions.

Proof: Write $\sigma \in \operatorname{Aut}(\pi)$ as $\sigma = (h, \Phi)$ with $h \in \operatorname{Hom}(N, K)$, $\Phi \in \operatorname{Aut}(N)$. Now $(0, \Phi) \circ (h, 1) = \left(\Phi'(h), \Phi\right) = \left(\Phi'(h), 1\right) \circ (0, \Phi)$. Hence $\left((0, \Phi) \circ (h, 1)\right)(\alpha, u)$ $= (0, \Phi)\left(\alpha + h(u), u\right) = \left(\Phi \wedge \Phi\left(\alpha + h(u)\right), (u)\right)$. But $\left(\left(\Phi'(h), 1\right) \circ (0, \Phi)\right)(\alpha, u)$ $= \left(\Phi'(h), 1\right)\left(\Phi \wedge \Phi(\alpha), \Phi(u)\right) = \left(\Phi \wedge \Phi(\alpha) + \Phi'(h)\left(\Phi(u)\right), \Phi(u)\right)$. Hence,

$$\Phi \wedge \Phi(h(u)) = \Phi'(h)(\Phi(u)), \quad \text{for all } u \in K,$$

which yields the lemma.

We turn our attention now to the outer automorphism group, $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$.

(3.4) Lemma: The map $N \to \mathrm{Hom}(N, K) \circ \mathrm{Aut}(N)$ given by $u \mapsto (h_u, 1)$, where $h_u(v) = u \wedge v$, is an injective homomorphism.

Proof: It is trivially injective, and trivially a homomorphism.

We regard N as a subgroup of $\mathrm{Hom}(N, K) \circ \mathrm{Aut}(N)$. Note that the action of $\mathrm{Aut}(N)$ is given by $\Phi'(h_u) = h_{\Phi(u)}$.

(3.5) Lemma: $\mathrm{Out}(\pi)$ is naturally isomorphic with $\mathrm{hom}(N, K) \circ \mathrm{Aut}(N)$, where $\mathrm{hom}(N, K) = \mathrm{Hom}(N, K)/2N$, regarded as subgroups of $\mathrm{Out}(\pi)$.

Proof: Since $(0, u)(\alpha, v)(0, -u) = (\alpha + 2u \wedge v, v) = h_{2u}(\alpha, v)$, $\mathrm{Inn}(\pi)$ is isomorphic with $2N$. Hence $\mathrm{Out}(\pi)$ is isomorphic with $\mathrm{Aut}(\pi)/2N$, and this is isomorphic with $\mathrm{hom}(N, K) \circ \mathrm{Aut}(N)$, since $(h, \Phi) \in 2N$ if and only if $\Phi = 1$ and $h = h_{2u}$ for some $u \in N$.

4. Φ, $\Phi \wedge \Phi$, Φ'

We have $\Phi \in \mathrm{Aut}(N) = GL(n, \mathbb{Z})$. Henceforth, we shall assume that the characteristic polynomial of Φ, $f(x) = \det(xI - \Phi)$, is irreducible over $\mathbb{Q}$, the field of rational numbers. Let the zeros of $f(x)$ in the complex numbers, $\mathbb{C}$, be $\lambda_1, \lambda_2, \ldots, \lambda_n$. By assumption, they are distinct.

Now, K is a free $\mathbb{Z}$-module of rank $k = n(n-1)/2$, and hence $K \otimes \mathbb{C}$ is a complex vector space of dimension k, and $(\Phi \wedge \Phi) \otimes 1$ is a linear map.

(4.1) Lemma: $(\Phi \wedge \Phi) \otimes 1$ is diagonalizable over $\mathbb{C}$, and the characteristic polynomial of $\Phi \wedge \Phi \in \mathrm{Aut}(K)$ is $\det(xI - \Phi \wedge \Phi)$

$$= \prod_{1 \le i < j \le n} (x - \lambda_i \lambda_j) \in \mathbb{Z}[x].$$

Proof: It suffices to show that $(\Phi \wedge \Phi) \otimes 1$ has k linearly independent eigenvectors with eigenvalues $\lambda_i \lambda_j$, $1 \le i < j \le n$. By assumption, $\Phi \otimes 1$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, so we may choose a $\mathbb{C}$-basis $\{u_j, 1 \le j \le n\}$ of $N \otimes \mathbb{C}$ such that $(\Phi \otimes 1)(u_j) = \lambda_j u_j$, $1 \le j \le n$. Then $\{u_i \wedge u_j, 1 \le i < j \le n\}$ is a $\mathbb{C}$-basis for $K \otimes \mathbb{C}$, and $((\Phi \wedge \Phi) \otimes 1)(u_i \wedge u_j) = \lambda_i \lambda_j u_i \wedge u_j$.

$\operatorname{Hom}(N, K)$ is a free $\mathbb{Z}$-module of rank nk. The induced action Φ' of Φ was given in Lemma 3. 3. There is a natural action $\Phi' \otimes 1$ on $\operatorname{Hom}(N \otimes \mathbb{C}, K \otimes \mathbb{C})$, an nk-dimensional complex vector space.

(4. 2) Lemma: $\Phi' \otimes 1$ is diagonalizable over $\mathbb{C}$ and the characteristic polynomial of Φ' is

$$\det(xI - \Phi') = \prod_{k=1}^{n} \prod_{1 \le i < j \le n} \left(x - (\lambda_i \lambda_j)/\lambda_k\right)$$

Proof: Let $\{u_j\}$ be as in (4. 1). Define a basis $\{h^k_{i,j} : 1 \le k \le n, 1 \le i < j \le n\}$ of $\operatorname{Hom}(N \otimes \mathbb{C}, K \otimes \mathbb{C})$ by

$$h^k_{i,j}(u_r) = \begin{cases} u_i \wedge u_j & \text{if } k = r \\ 0 & \text{otherwise} \end{cases}$$

Then $(\Phi' \otimes 1)(h^k_{i,j}) = \lambda_i \lambda_j \lambda_k^{-1} h^k_{i,j}$. This yields the lemma.

5. Centralizers in $\operatorname{Out}(\pi)$

We have $\operatorname{Out}(\pi) = \operatorname{hom}(N, K) \circ \operatorname{Aut}(N)$. We shall use letters x, y, z to denote elements of $\operatorname{hom}(N, K)$, and write the induced action of Φ on $x \in \operatorname{hom}(N, K)$ as $\Phi'(x)$. Let $C(\Phi)$ be the centralizer of Φ in $\operatorname{Aut}(N)$. Let $N(\Phi) = \{\Psi \in \operatorname{Aut}(N) : \Psi \Phi \Psi^{-1} = \Phi^{\pm 1}\}$. Similarly, write $C(x, \Phi)$, $N(x, \Phi)$ for $(x, \Phi) \in \operatorname{Out}(\pi)$. Clearly there are homomorphisms $C(x, \Phi) \longrightarrow C(\Phi)$, $N(x, \Phi) \longrightarrow N(\Phi)$. Remember that $f(x) = \det(xI - \Phi)$ has distinct roots, $\lambda_1, \ldots, \lambda_n$, and is irreducible.

(5. 1) Lemma: Suppose that $(x, \Phi) \in \operatorname{Out}(\pi)$ and that $f(x) \ne x^n f(1/x)/f(0)$. Then $C(x, \Phi) = N(x, \Phi)$.

Proof: It suffices to show that $C(\Phi) = N(\Phi)$. Suppose not, so $\Psi\Phi\Psi^{-1} = \Phi^{-1}$. The characteristic polynomials are $f(x)$, $\prod_{i=1}^{n}(x - \lambda_i^{-1}) = (-1)^n x^{-n}(\prod\lambda_i)^{-1}\prod(1/x - \lambda_i) = x^n f(1/x)/f(0)$, which is impossible.

(5.2) Lemma: Suppose $(x, \Phi) \in Out(\pi)$, $f(1) = \det(I - \Phi) = \pm 1$ and that $\lambda_k \neq \lambda_i\lambda_j$, for all i, j, k, $i \neq j$. Then $\hom(N, K) \cap C(x, \Phi) = \{(0, I)\}$. Thus $C(x, \Phi) \longrightarrow C(\Phi)$ is injective.

Proof: If $(y, \Psi) \in C(x, \Phi)$ then $\Psi \in C(\Phi)$ and $(y, \Psi) \circ (x, \Phi) = (y + \Psi'(x), \Psi \circ \Phi) = (x, \Phi) \circ (y, \Psi) = (x + \Phi'(y), \Phi \circ \Psi)$. Thus,

5.3 $$(I - \Phi')(y) = (I - \Psi')(x)$$

In the case at hand, if (y, Ψ) is in the intersection then $\Psi' = I$ and so $y \in \mathrm{Ker}(I - \Phi' : \hom(N, K) \longrightarrow \hom(N, K))$. Let h be a representative for y in $\mathrm{Hom}(N, K)$. Then $(I - \Phi')(h) = h_{2u}$ for some $u \in N$. Since $\det(I - \Phi) = \pm 1$, there is a $w \in N$ such that $w - \Phi(w) = u$. Then it is easily seen that $h - h_{2w} \in \mathrm{Ker}(I - \Phi' : \mathrm{Hom}(N, K) \longrightarrow \mathrm{Hom}(N, K))$, but by 4.2 and our hypothesis $\mathrm{Ker}(I - \Phi')$ is trivial. Hence $h = h_{2w}$ and so $y = 0$ in $\hom(N, K)$. This completes the proof.

The next lemma follows immediately from 5.3.

(5.4) Lemma: If $(x, \Phi) \in Out(\pi)$ and $2x \notin \mathrm{Image}(I - \Phi')$ then $C(x, \Phi)$ contains no elements of the form $(y, -I)$.

The hypothesis of Lemma 5.4 can never hold when $n = 2$. To see this, let $N = \langle e_1 \rangle \oplus \langle e_2 \rangle$, $K = \langle e_1 \wedge e_2 \rangle$. Now $\mathrm{Hom}(N, K) = \langle h_1 \rangle \oplus \langle h_2 \rangle$, where

$$h_i(e_j) = \begin{cases} e_1 \wedge e_2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then, $2N = \langle 2h_1 \rangle \oplus \langle 2h_2 \rangle$, and therefore $\hom(N, K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ which contains no non-zero element of the form $2x$.

The next lemma provides a usable condition for the last lemma to be applicable.

(5.5) Lemma: If $\det(I - \Phi')$ is odd and not ± 1, and if $f(1) = \pm 1$, then there is an $x \in \operatorname{Im}\left(I - \Phi' : \hom(N, K)\right)$ such that $2x \notin \operatorname{Im}\left(I - \Phi' : \hom(N, K) \longrightarrow \hom(N, K)\right)$.

Proof: Write $\operatorname{Hom}(N, K) = N \oplus B$, where B is a free $\mathbb{Z}$-module of rank $nk - n$. By the remark before Lemma 3.5, Φ' maps N to N, so $I - \Phi'$ does also. Write elements of $N \oplus B$ as ordered pairs (a, b). Then $(I - \Phi')(a, b) = \left(\sigma(a) + \tau_1(b), \tau(b)\right)$, where σ, τ_1, τ are the obvious induced linear maps. If $\alpha \in N/2N$, then $(I - \Phi')(\alpha, b) = \left(\left(\sigma(a) + \tau_1(b)\right) (\text{mod } 2N), \tau(b)\right)$. Hence, if $x = (\alpha_1, b_1)$ then $2x = (0, 2b_1) \notin \operatorname{Im}\left(I - \Phi' : \hom(N, K) \longrightarrow \hom(N, K)\right)$ if and only if $2b_1 \notin \operatorname{Im}(\tau : B \to B)$. Now $\det(I - \Phi') = \det(\sigma)\det(\tau)$. But clearly $\sigma(a) = a - \Phi(a)$, so $\det(\sigma) = \det(I - \Phi) = \pm 1$. If $\det(\tau)$ is odd, but not ± 1, then $B/\operatorname{Im}(\tau)$ is a non-trivial group of odd order, and so contains a non-zero element of the form $2b_1$.

6. The group L

For $(x, \Phi) \in \operatorname{Out}\pi = \hom(N, K) \circ GL(n, Z)$ we choose a representative automorphism $\sigma_h \circ \sigma_\Phi : \pi \longrightarrow \pi$. By following the procedure in [2] we use this automorphism to introduce the semi-direct product $L = \pi \circ \mathbb{Z}$. We shall determine $\operatorname{Out}(L)$ in many cases by using results from [2].

(6.1) Lemma: Suppose that $\Phi \in GL(n, Z)$ has infinite order and that $\det(I - \Phi) \cdot \det(I - \Phi \wedge \Phi) \neq 0$. Then,

(i) π is a characteristic subgroup of L,

(ii) L has trivial center

(iii) the sequence

$$0 \to K \xrightarrow{I - \Phi \wedge \Phi} K \longrightarrow \operatorname{Out}(L) \longrightarrow N(x, \Phi)/(x, \Phi) \longrightarrow 1$$

is exact.

Proof: First we note that the commutator subgroup $[\pi, \pi] = 2K$. Therefore

$$(\pi/[\pi, \pi]) \otimes Q \cong N \otimes Q$$

and $\sigma_h \circ \sigma_\Phi$ induces $\Phi \otimes 1$ on $N \otimes Q = H_1(\pi; Q)$. Since $\det(I - \Phi) \neq 0$, we can apply [2; 4.1] to see that π is a characteristic subgroup of L. Next observe

that the restriction of $\sigma_h \circ \sigma_\Phi$ to K, the center of π, is $K \xrightarrow{\Phi \wedge \Phi} K$ since $\sigma_h \circ \sigma_\Phi(\alpha, 0) = (\Phi \wedge \Phi(\alpha), 0)$. But $\det(I - \Phi \wedge \Phi) \neq 0$ and (x, Φ) has infinite order in $\mathrm{Out}(\pi)$. Therefore it follows from [2; 4.7] that L has a trivial center.

Now, [2; 4.6] yields

$$0 \longrightarrow H_0(Z; K) \longrightarrow \mathrm{Out}(L) \longrightarrow N(x, \Phi)/(x, \Phi) \longrightarrow 1$$

which is the same statement as (iii) of the lemma.

Let us now assume that the hypotheses of Section 2 hold, namely that the characteristic polynomial of Φ, $f(t) = \det(tI - \Phi)$, is irreducible over $\mathbb{Q}$ and satisfies (2.8). Then $C(x, \Phi) = N(x, \Phi)$ by 5.1. To now choose x we assume the hypothesis of 5.5.

(6.2) Theorem: Let the characteristic polynomial of $\Phi \in GL(n, \mathbb{Z})$ be irreducible over $\mathbb{Q}$ and assume that Φ satisfies:

(i) (2.1), . . . , (2.5)

(ii) $\det(I - \Phi \wedge \Phi) = \pm 1$

(iii) $\det(I - \Phi')$ is odd but not ± 1.

Then, $x \in \hom(N, K)$ can be chosen so that the group $L = \pi \circ_{(x, \Phi)} \mathbb{Z}$ has trivial center and

$$\mathrm{Out}(L) \cong C(x, \Phi)/(x, \Phi)$$

has no elements of finite order.

Proof: We note first that Φ is of infinite order. The centralizer $C(\Phi) \subset GL(n, \mathbb{Z})$ is isomorphic with a subgroup of the (abelian) group of units of the ring of integers in $\mathbb{Q}(\lambda)$ where λ is some root of $f(t) = \det(tI - \Phi)$. The conditions (2.1), . . . , (2.5) imply

(a) $C(\Phi) = N(\Phi) \subset GL(n, \mathbb{Z})$ is abelian,

(b) the infinite cyclic subgroup generated by Φ is a direct summand of $C(\Phi)$

(c) $-I$ is the only element of finite order in $C(\Phi)$.

Since $N(\Phi) = C(\Phi)$ we have that $C(x, \Phi) = N(x, \Phi)$. From 5.2, since (iii) holds, we have $C(x, \Phi) \longrightarrow C(\Phi)$ is a monomorphism. Then this image does not contain $-I$ by 5.4 and 5.5. Hence it follows that $C(x, \Phi)$ contains no non-trivial elements of finite order.

Suppose now that $(y, \Psi) \in C(x, \Phi)$ and that $(y, \Psi)^n \in ((x, \Phi))$, the infinite cyclic subgroup generated by (x, Φ). Then Ψ^n lies in the infinite cyclic subgroup of $C(\Phi)$ generated by Φ. But this is a direct summand of $C(\Phi)$. Thus $\Psi = \Phi^k$ for some integer k. Now (y, Φ^k) and $(x, \Phi)^k$ have the same image under $C(x, \Phi) \longrightarrow C(\Phi)$. Since this is a monomorphism $(x, \Phi)^k = (y, \Phi^k)$. Thus $C(x, \Phi)/((x, \Phi))$ contains no elements of finite order. By (ii) $H_0(Z; K) = 0$ and thus $C(x, \Phi)/((x, \Phi)) \cong \mathrm{Out}(L)$.

(6. 3) Corollary: *Suppose, in addition,* $C(\Phi)$ *has rank* 1. *Then* Out(L) *is trivial.*

7. Some matrices satisfying the hypotheses of §6

In this section we present some matrices $\Phi \in GL(n, \mathbf{Z})$ which satisfy the hypotheses of 6. 2. These matrices will be used in §9 to construct closed aspherical manifolds in dimensions 7, 11, 16, 22, 29, and 37 which admit no periodic maps. In §2 these matrices are determined by conditions on their characteristic polynomials and eigenvalues. If the polynomial is $f(x) = \prod_{i=1}^n (x - \lambda_i)$, then the new conditions in addition to those of §2 (2. 6, 2. 7, 2.8, 2. 10, 2. 11) are:

7. 1 $$\prod_{1 \leq i < j \leq n} (1 - \lambda_i \lambda_j) = \det(I - \Phi \wedge \Phi) = \pm 1$$

from (ii) of 6. 2, and

7. 2 $$\prod_{k=1}^n \prod_{1 \leq i < j \leq n} (1 - \lambda_i \lambda_j \lambda_k^{-1}) = \det(I - \Phi')$$

is odd but not ± 1, from (iii) of 6. 2 (or Lemma 5. 5).

For each example we give the polynomial, its roots, $\det \Phi$, $\det(I - \Phi \wedge \Phi)$ and $\det(I - \Phi')$.

Example: 1. $M^7 : f(x) = x^3 + x - 1$, $\lambda_1 = .682378$, $\lambda_2, \lambda_3 = -.3411639 \pm 1.1615414i$, $\det(\Phi) = 1$, $\det(I - \Phi \wedge \Phi) = -1$, $\det(I - \Phi') = -3$.

Example 2. $M^{11} : f(x) = x^4 - x + 1$, $\lambda_1, \lambda_2 = .727136 \pm .4300143i$, $\lambda_3, \lambda_4 = -.7271361 \pm .9340993i$, $\det(\Phi) = 1$, $\det(I - \Phi \wedge \Phi) = -1$, $\det(I - \Phi') = 47$.

Example 3. $M^{16}: f(x) = x^5 - x^2 + 1$, $\lambda_1 = -.8087306$, $\lambda_2, \lambda_3 = .8692775 \pm .3882694i$, $\lambda_4, \lambda_5 = -.4649122 \pm 1.0714738i$, $\det \Phi = -1$, $\det(I - \Phi \wedge \Phi) = -1$, $\det(I - \Phi') = 859$.

Example 4. $M^{22}: f(x) = x^6 + x - 1$, $\lambda_1 = .7780896$, $\lambda_2 = -1.134724$, $\lambda_3, \lambda_4 = -.4510552 \pm 1.0023646i$, $\lambda_5, \lambda_6 = .6293724 \pm .7357560i$, $\det \Phi = -1$, $\det(I - \Phi \wedge \Phi) = -1$, $\det(I - \Phi') = 3299$.

Example 5. $M^{29}: f(x) = x^7 + x^3 - 1$, $\lambda_1 = .8631465$, $\lambda_2, \lambda_3 = -.8717349 \pm .5787134i$, $\lambda_4, \lambda_5 = -.3074645 \pm .8580940i$, $\lambda_6, \lambda_7 = .7476262 \pm .8453860i$, $\det \Phi = 1$, $\det(I - \Phi \wedge \Phi) = 1$, $\det(I - \Phi') = 593$.

Example 6. $M^{37}: f(x) = x^8 - x^3 - 1$, $\lambda_1 = -.8724946$, $\lambda_2 = 1.114798$, $\lambda_3, \lambda_4 = .5755985 \pm .7073086i$, $\lambda_5, \lambda_6 = .1288103 \pm 1.0097414i$, $\lambda_7, \lambda_8 = -.8255605 \pm .7152643i$, $\det \Phi = -1$, $\det(I - \Phi \wedge \Phi) = 1$, $\det(I - \Phi') = -89$.

All the polynomials given above satisfy the conditions (2.6), (2.7), (2.8), (2.10), (2.11), given in Section 2. The first three conditions are trivial, Ljunggren's theorem applies to all of them, and none can contain a complex root of one. This last statement is true since each polynomial must have a real root, except the one for M^{11}. However, for M^{11}, the argument at the end of Section 2 applies.

The examples were constructed by taking polynomials satisfying the conditions of Section 2, calculating their roots, and then calculating the two products at the beginning of this section. The computations were done in long precision, about 15 significant digits, on an IBM 360. The roots given above were rounded to seven decimal places. The calculated values for $\det(I - \Phi \wedge \Phi)$, $\det(I - \Phi')$ differed from the integers given by less than 10^{-8}.

The conditions of Section 2, together with (7.1) and (7.2) form a seemingly restrictive set of conditions. It is not clear to us whether this sequence of examples can be continued indefinitely. Taking $f(x) = x^3 - x + 1$ would also have worked, but $f(x) = x^3 - x - 1$ has $\det(I - \Phi') = 1$. It is easy to prove that $\det(I - \Phi \wedge \Phi)$ will be even if $f(x)$ has a quadratic factor (mod 2). This is the same as the condition $n_1 + m_1 \equiv 0 \pmod 3$ in Ljunggren's theorem.

Notice that Examples 1 and 2 also satisfy Corollary 6.3.

8. Aspherical manifolds which admit only involutions.

Let $\Phi \in GL(n, Z)$ be a matrix of the form

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & \cdots & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ \cdot & & & & 0 & 1 & 0 \\ 0 & \cdots & & & 0 & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & & & & -a_1 \end{bmatrix}$$

Here Φ is the companion matrix of its characteristic polynomial

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \quad .$$

where $a_i \in \mathbb{Z}$.

We know that

$$f(x) = \det(xI - \Phi) \ ,$$

and hence

$$f(0) = (-1)^n \det \Phi = a_n \ ,$$

that is,

$$\det \Phi = (-1)^n a_n \quad ,$$

and

$$f(1) = \det(I - \Phi) \ .$$

Let $T^n = \{(z_1, \ldots, z_n)\}$. We define a homeomorphism $h: (T^n, 1) \longrightarrow (T^n, 1)$ where 1 is short for $(1, \ldots, 1) \in T^n$, by

$$h(z_1, z_2, \ldots, z_n) = (z_n^{-a_n}, z_1 z_n^{-a_{n-1}}, \ldots, z_{n-2} z_n^{-a_2}, z_{n-1} z_n^{-a_1})$$

Now $h_*: \pi_1(T^n, 1) \longrightarrow \pi_1(T^n, 1)$ is precisely our matrix Φ.

We form $N^{n+1}(\Phi) = \mathbb{R}^1 \times_{\mathbb{Z}} T^n$ by defining a free action of $\mathbb{Z}$ on $\mathbb{R}^1 \times T^n$

$$(r; z_1, \ldots, z_n) \longrightarrow \big(r - k; h^k(z_1, z_2, \ldots, z_n)\big) \ .$$

The quotient manifold N^{n+1} fibers over the circle $\mathbb{R}^1/\mathbb{Z}$ with fiber T^n and structure group generated by h.

The fundamental group $\pi_1(N^{n+1})$ is a semi-direct product $Z^n \circ Z$ with group law given by $(\alpha, i)(\beta, j) = \left(\alpha + \Phi^i(\beta),\ i+j\right)$. We may present the group as follows:

$$\pi_1(N^{n+1}, *) = \left\{ x_1, \dots, x_n, h : hx_1h^{-1} = x_n^{-a_n},\ hx_ih^{-1} = x_{i-1}x_n^{-a_{n-i+1}} \right.$$
$$\left. \text{for } i = 2, 3 \dots n,\ [x_k, x_\ell] = 1, \text{ for all } k \text{ and } \ell \right\}.$$

Notice there is an obvious "standard" action of Z_2 on N^{n+1} given by sending

$$((r; z_1, z_2, \dots, z_n)) \longrightarrow ((r; z_1^{-1}, z_2^{-1}, \dots, z_n^{-1})).$$

It is well defined because

$$\begin{array}{ccc}
(r; z_1, z_2, \dots, z_n) & \longrightarrow & (r; z_1^{-1}, z_2^{-1}, \dots, z_n^{-1}) \\
\downarrow & & \downarrow \\
(r-1; z_n^{-a_n}, z_1 z_n^{-a_{n-1}}, \dots, z_{n-1} z_n^{-a_1}) & \longrightarrow & (r-1; z_n^{a_n}, z_1^{-1} z_n^{a_{n-1}}, \dots, z_{n-1}^{-1} z_n^{a_1})
\end{array}$$

commutes. Notice, also, that the fixed point set is a disjoint collection of circles.

(8.2) Theorem: Let $N^{n+1}(\Phi) = \mathbb{R}^1 \times_Z T^n$ be constructed as above where Φ is the companion matrix in $GL(n, Z)$ to any one of the irreducible polynomials described in 2.12. Then any non-trivial action $\left(G, N^{n+1}(\Phi)\right)$ of a finite group implies that $G \cong Z_2$.

Furthermore, $F\left(Z_2, N^{n+1}(\Phi)\right)$ is a non-empty disjoint collection of circles. In particular, for each $n+1 > 3$, $N^{n+1}(\Phi)$ can be chosen orientable. Consequently, in this case, G must preserve orientation if $n+1$ is odd and reverse orientation if $n+1$ is even.

Proof: Each of the polynomials from the list in 2.12 satisfies (2.6), (2.7), (2.8), (2.10) and (2.11). As we have seen in 6.2 the automorphism

$$\Phi: \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$$

guarantees us that

$$\mathbb{Z}^n \circ \mathbb{Z} \cong \pi_1\left(N^{n+1}(\Phi), *\right)$$

is a centerless group as well as satisfying (a), (b) and (c) there. Thus the normalizer of Φ is the centralizer of Φ, $C(\Phi)$, and $C(\Phi)/(\Phi) \cong \mathbb{Z}^r \oplus Z_2$. The integer r satisfies $r \leq r_1 + \frac{n - r_1}{2} - 1$ where r_1 is the number of real roots of $f(x)$. Since $\det(I - \Phi) = \pm 1 = f(1)$ we know that $H_0(\mathbb{Z}; \mathbb{Z}^k) = 0$ and by [2; 4.7]

$$\mathrm{Out}\,\pi_1\left(N^{n+1}(\Phi), *\right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^r \ .$$

By Borel's theorem [2; 3.2] any effective group action $\left(G, N(\Phi)\right)$ implies $G \cong \mathbb{Z}_2$ if G is non-trivial and finite. Since there exists exactly one embedding of $\mathbb{Z}_2$ in Out(L), Theorem 7.12 of [2] and the Appendix of [2] imply that $F\left(G, N^{n+1}(\Phi)\right)$ is homeomorphic to the disjoint collection of circles which form the fixed point set of the "standard" Z_2 action.

When $n+1 = 3$ or 4 it is possible to choose $f(x)$ so that $C(\Phi) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Thus, $\mathrm{Out}(L) \cong \mathbb{Z}_2$ in these cases.

Of course it is not necessary to choose Φ to be the companion matrix for $f(x)$. Any matrix Φ whose characteristic polynomial is $f(x)$ will suffice. The advantage of choosing Φ to be the companion matrix is that $N^{n+1}(\Phi)$ has such an explicit and simple form as well as our being able to exhibit an explicit Z_2-action on N^{n+1}.

9. Closed aspherical manifolds without periodic maps

It just remains to show that we may topologically realize the construction of §7. Let π be as in §3.

(9.1) Theorem: If L is the semi-direct product $\pi \circ \mathbb{Z}$ defined by an automorphism $\sigma_h \circ \sigma_\Phi : \pi \longrightarrow \pi$ then there is a closed connected aspherical $((n^2 + n + 2)/2)$-dimensional manifold M for which $\pi_1(M) \cong L$.

Proof: Let $V = \mathbb{R}^n$ so that $N \subset V$ and $\Lambda^2(V) \supset K$. Then on the cartesian product $G = \Lambda^2(V) \times V$ we may introduce a Lie group operation by

$$(\vec{\alpha}, \vec{u}) \cdot (\vec{\beta}, \vec{v}) = (\vec{\alpha} + \vec{\beta} + \vec{u} \wedge \vec{v}, \vec{u} + \vec{v}) .$$

We use arrows over the symbols to denote vectors in V and $\Lambda^2(V)$ respectively. Then G is a Lie group structure on R^{k+n}, $k=(n(n-1))/2$, and $\pi \subset G$ is a discrete subgroup for which G/π is a compact, aspherical quotient manifold.

Now an element $(h, \Phi) \in \mathrm{Hom}(N, K) \circ GL(n, Z) = \mathrm{Aut}(\pi)$ may be regarded as an automorphism of G which leaves π invariant. We simply extend h uniquely to a real linear transformation $h: V \longrightarrow \Lambda^2(V)$ and observe that $GL(n, Z) \subset GL(n, R)$ so that the automorphism of G is

$$(\vec{\alpha}, \vec{u}) = \left(\Phi \wedge \Phi(\vec{\alpha}) + h(\Phi(\vec{u})), \Phi(\vec{u})\right) .$$

This in turn induces a homeomorphism $\Psi: G/\pi \longrightarrow G/\pi$ which leaves the point $e \in G/\pi$ fixed. Of course $\pi_1(G/\pi, e) \cong \pi$ and $\Psi_*: \pi_1(G/\pi, e) \cong \pi_1(G/\pi, e)$ is (h, Φ).

To finish we need only take $I \times G/\pi$ and make the identification $(0, y) \sim (1, \Psi(y))$. The resulting M then has $\pi_1(M) \cong L$ and R^{k+n+1} as the universal covering space.

If (x, Φ) is as in §6. 2, then $\mathrm{Out}(\pi_1(M))$ has no elements of finite order and hence by Borel's theorem [2; 3. 2] M has no non-trivial periodic homeomorphisms. If 6. 3 is applicable, then $\mathrm{Out}(\pi_1(M))$ is completely trivial, which is the case in Examples 1 and 2 of §7.

10. Homogeneous spaces

From a small additional consideration it is possible to produce examples which are the quotients of a solvable Lie group structure on R^7 by discrete uniform subgroups. (The theory, of course, is not restricted to R^7 but we have found examples only in this dimension.)

We introduced the nilpotent Lie group $G = \Lambda^2(V) \times V$ in the last section. Let $\mathrm{Hom}_R(V, \Lambda^2(V))$ be the vector space of real linear maps of V into $\Lambda^2(V)$, then, by an obvious analogy, a left action of $SL(3, R)$ on $\mathrm{Hom}_R(V, \Lambda^2(V))$ is introduced with $\Phi'(\ell)(\vec{u}) \equiv \Phi \wedge \Phi(\ell(\Phi^{-1}(\vec{u})))$. The semi-direct product $\mathrm{Hom}_{\mathbb{R}}(V, \Lambda^2(V)) \circ SL(3, \mathbb{R})$ contains $\mathrm{Aut}(\pi) = \mathrm{Hom}(N, K) \circ SL(3, Z)$ and acts from the left on G as a group of automorphisms.

Suppose now that $\Phi \in SL(3, \mathbb{Z})$, $\det(I - \Phi) = \pm 1$ and that Φ lies on a 1-parameter subgroup of $SL(3, \mathbb{R})$, $t \to \Phi_t$, with $\Phi_1 = \Phi$. We may assert, then, that for any homomorphism $h: N \longrightarrow K$ the pair (h, Φ) lies on a 1-parameter

subgroup of $\mathrm{Hom}_{\mathbb{R}}\left(V, \Lambda^2(V)\right) \circ SL(3, \mathbb{R})$. We first note that by 2.3 it follows that $\ell \longrightarrow \ell - \Phi'(\ell)$ is non-singular on $\mathrm{Hom}_{\mathbb{R}}\left(V, \Lambda^2(V)\right)$ and thus for a unique $\ell : V \longrightarrow \Lambda^2(V)$ we have $\Phi'(\ell) - \ell = h$. Now we set

$$e(t) = \left(\Phi_t'(\ell) - \ell, \Phi_t\right)$$

in $\mathrm{Hom}_{\mathbb{R}}\left(V, \Lambda^2(V)\right) \circ SL(3, \mathbb{R})$. Noting that $\left(\Phi_{t_1}'(\ell) - \ell, \Phi_{t_1}\right)\left(\Phi_{t_2}'(\ell) - \ell, \Phi_{t_2}\right)$ $= \Phi_{t_1}'(\ell) - \ell + \Phi_{t_1}'\left(\Phi_{t_2}'(\ell)\right) - \Phi_{t_1}'(\ell) = \Phi_{t_1+t_2}'(\ell) - \ell$, we see that $e(t)$ is the 1-parameter subgroup with $e(1) = (h, \Phi)$.

But, using this $e(t)$, we may introduce the semi-direct product, $H = G \circ \mathbb{R}$, of G with the reals. Obviously $L = \pi \circ Z \subset H$ as a discrete subgroup and H/L is a compact connected 7-manifold of the preceding section.

R. J. Koch has pointed out the matrix

$$\Phi = \begin{pmatrix} 5 & -7 & 6 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in $SL(3, Z)$. This satisfies (i), (ii), and (iii) of 6.2. In addition, the eigenvalues of Φ are real and positive so that Φ does lie on a 1-parameter subgroup of $SL(3, \mathbb{R})$. In this way we find that, for a suitable selection of h, <u>the solvmanifold H/L admits no periodic homeomorphisms other than the identity.</u>

11. <u>Fundamental matrices in</u> $GL(3, Z)$

In this section we shall give two infinite families of distinct irreducible polynomials of degree 3 which gives rise to examples having no periodic homeomorphisms and which are also closed orientable 7-dimensional solvmanifolds. In fact we partially determine the characteristic polynomials of all the matrices in $GL(3, \mathbb{Z})$ which will satisfy conditions of (i), (ii) and (iii) of 6.2. Explicit matrix representations in $GL(3, \mathbb{Z})$ can be easily obtained by choosing the companion matrix for such a polynomial in the manner of §8.

It will be readily seen that the difficulty, and this is also true for $n > 3$, lies in checking (2.11).

For $\Phi \in GL(3, \mathbb{Z})$ let

$$f(x, \Phi) = x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$$

be its characteristic polynomial. Suppose the eigenvalues of Φ are $\lambda_1, \lambda_2, \lambda_3$. Now $\Phi \wedge \Phi \in GL(3, \mathbb{Z})$, and its easy to see that its characteristic polynomial is

$$f(x, \Phi\wedge\Phi) = x^3 - \sigma_2 x^2 + \sigma_1 \sigma_3 x - \sigma_3^2 \quad .$$

Also $\Phi' \in GL(9, \mathbb{Z})$ and its characteristic polynomial is

$$f(x, \Phi') = f(x)^2 \left(x^3 - \frac{\sigma_2^2 - 2\sigma_3\sigma_1}{\sigma_3} x^2 + (\sigma_1^2 - 2\sigma_2)x - \sigma_3\right).$$

We require that

11. 1 $\qquad f(0, \Phi) = \pm 1$

11. 2 $\qquad f(1, \Phi) = \pm 1$

11. 3 $\qquad f(1, \Phi \wedge \Phi) = \pm 1$

11. 4 $\qquad f(1, \Phi')$ be odd, but not ± 1

11. 5 $\qquad f(x^k, \Phi)$ have no cubic factor when k is prime.

The first two conditions give four cases which may be summarized as:

$\sigma_3 = 1, \ \sigma_2 = \sigma_1 + 1 \Longrightarrow f(1, \Phi\wedge\Phi) = -1, \ f(1, \Phi') = -2\sigma_1 - 3$, and $\neq \pm 1$

iff $\sigma_1 \neq -1, -2.$

$\sigma_3 = 1, \ \sigma_2 = \sigma_1 - 1 \Longrightarrow f(1, \Phi\wedge\Phi) = 1, \ f(1, \Phi') = 2\sigma_1 + 1$ and $\neq \pm 1$

iff $\sigma_1 \neq 0, -1$

$\sigma_3 = -1, \ \sigma_2 = \sigma_1 - 1 \Longrightarrow f(1, \Phi\wedge\Phi) = -2\sigma_1 + 1 = \pm 1$ iff $\sigma_1 = 0, 1$

$\Longrightarrow f(1, \Phi') = 1$ if $\sigma_1 = 0$

5 if $\sigma_1 = 1$

$$\sigma_3 = -1,\ \sigma_2 = \sigma_1 - 3 \Longrightarrow f(1, \Phi\wedge\Phi) = 3 - 2\sigma_1 = \pm 1 \text{ iff } \sigma_1 = 1, 2$$

$$\Longrightarrow f(1, \Phi') = 1 \text{ if } \sigma_1 = 1$$

$$3 \text{ if } \sigma_1 = 2$$

Incidently, note that σ_1 is trace (Φ) and in particular these conditions tell which traces must be avoided.

We consider the first two cases together. Let k be an odd prime and suppose

$$g(x) = x^3 - b_1x^2 + b_2x - b_3$$

is a factor of

$$h(x) = x^{3k} - \sigma_1 x^{2k} + \sigma_2 x^k - 1$$

Suppose the roots of $g(x)$ are $\beta_1, \beta_2, \beta_3$ with $\beta_i^k = \lambda_i$. Then

$$(\beta_1 + \beta_2 + \beta_3)^k = (\beta_1^k + \beta_2^k + \beta_3^k) + k \sum_{\substack{0 \leq i, j, m \\ i+j+m = k}} \frac{\beta_1^i \beta_2^j \beta_3^m (k-1)!}{i!\ j!\ m!} \ .$$

The left hand side is b_1^k and the first term on the right hand side is σ_1. It is easy to see that each term in the summation has integer coefficients and so the summation is a polynomial in b_1, b_2, b_3 with integral coefficients (fundamental theorem on symmetric functions). Hence $b_1^k \equiv \sigma_1$ (mod k). Since k is prime, $b_1^{k-1} \equiv 1$ (mod k), so $b_1 \equiv \sigma_1$ (mod k). A similar argument shows that $b_2 \equiv \sigma_2$ (mod k). Since $\lambda_1\lambda_2\lambda_3 = 1$, $\beta_1\beta_2\beta_3 = 1$, hence $b_3 = 1$. Because the quotient polynomial has integral coefficients, $g(1)$ divides $h(1)$. Since $h(1) = f(1, \Phi)$, this means $(1 - b_1 + b_2 - b_3)$ divides ± 1. Hence $b_2 - b_1 = \pm 1$. But $b_2 - b_1 \equiv \sigma_2 - \sigma_1$ (mod k), so in our two cases,

$$\underline{\text{if}}\ \sigma_2 = \sigma_1 + 1 \qquad \underline{\text{then}} \quad b_2 = b_1 + 1$$

$$\underline{\text{if}}\ \sigma_2 = \sigma_1 - 1 \qquad \underline{\text{then}} \quad b_2 = b_1 - 1$$

since $k \geq 3$.

We now derive a necessary condition for $g(x)$ to divide $h(x)$.

(11.6) Lemma: *If* $g(x)$ *divides* $h(x)$ *with* g *and* h *as above, then*

$$\frac{\operatorname{disc}(f)}{\operatorname{disc}(g)}$$

is the square of an integer.

Proof: The discriminants are

$$\operatorname{disc}(g) = \prod_{1 \le i < j \le 3} (\beta_i - \beta_j)^2$$

$$\operatorname{disc}(f) = \prod_{1 \le i < j \le 3} (\beta_i^k - \beta_j^k)^2 .$$

Hence

$$\frac{\operatorname{disc}(f)}{\operatorname{disc}(g)} = \left[\prod_{1 \le i < j \le 3} (\beta_i^{k-1} + \beta_i^{k-2}\beta_j + \ldots + \beta_j^{k-1}) \right]^2 .$$

This expression inside the brackets is a symmetric function of β_1, β_2 and β_3 and therefore is an integer.

From [5; p. 83]

$$\operatorname{disc}(x^3 + a_1x^2 + a_2x + a_3) = a_1^2a_2^2 - 4a_2^3 - 4a_1^3a_3 - 27a_3^2 + 18a_1a_2a_3 .$$

We shall treat in detail only the case where $\sigma_2 = \sigma_1 + 1$. Then

$$f(x, \Phi) = x^3 - ax^2 + (a+1)x - 1, \quad \text{with } a \ne -1, -2.$$

And

$$\operatorname{disc}\bigl(f(x)\bigr) = a^4 - 6a^3 + 7a^2 + 6a - 31 .$$

Since $g(x) = x^3 - bx^2 + (b+1)x - 1$, the condition $g(-1)$ divides $h(-1)$ yields

$$2b + 3 \ \underline{\text{divides}}\ 2a + 3 .$$

Now suppose $2a + 3 = p$ is prime (positive or negative). The possible values of b are $a, -1, -2, -a-3$. If $b = a$, then $\beta_1 = \lambda$, so $k = 1$.

For the other three possibilities we use the following table, where $\Delta(a) = \operatorname{disc}\bigl(f(x)\bigr)$:

a	-2	-1	0	1	2	3	4	5
$\Delta(a)$	49	-23	-31	-23	-23	-32	-215	49

All the values of a for which $\Delta(a)$ is less than 0 are given. If $b = -1$, then we must have

$$\frac{\Delta(a)}{-23} = \text{square} \Longrightarrow a = -1, 1, 2$$

If $b = -2$, we must have

$$\frac{\Delta(a)}{49} = \text{square, so } \Delta(a) = \text{square} .$$

All such values of a can, in principle, be found. However, we limit ourselves to the observations that when we reduce

$$\text{mod } 5: \ \Delta(a) \not\equiv \text{square}, \quad \text{if } a \equiv 1, 2, 3 \pmod 5$$

$$\text{mod } 7: \ \Delta(a) \not\equiv \text{square}, \quad \text{if } a \equiv 1, 2, 3, 6 \pmod 7 .$$

If $b = -a-3$, then we must have $k = |2a+3|$ since $b \equiv a \pmod k$. But we show that this is impossible. Since

$$(|a|+2)^3 > |a|(|a|+2)^2 + (|a|+1)(|a|+2) + 1$$

it follows that

$$|\lambda_j| < (|a|+2) .$$

Because $k \geq 3$, $|b| = |a+3| < 3(|a|+2)^{1/3}$ which is false for large enough $|a|$.

Let us now look at $k = 2$. Then

$$x^3 - b_1x^2 + b_2x - 1 \quad \text{divides} \quad x^6 - ax^4 + (a+1)x^2 - 1 .$$

We have

$$b_1^2 = (\beta_1+\beta_2+\beta_3)^2 = a + 2b_2$$

$$b_2^2 = (\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)^2 = a + 1 + 2b_1$$

Now $b_2 - b_1 = \pm 1$. If

$$b_2 = b_1 + 1, \quad \text{then} \quad b_1^2 = a + 2b_1 + 2$$

$$b_1^2 = a$$

which implies

$$b_1 = -1, \quad b_2 = -2, \quad a = 1 .$$

In any case, if $x^3 - b_1x^2 + b_2x - 1$ does divide $x^6 - ax^4 + (a+1)x^2 - 1$, it is easy to see that the quotient is $x^3 + b_1x^2 + b_2x + 1$. These then are the only possibilities with $k = 2$.

In summary we have

(11.7) Theorem: The irreducible polynomials

$$f(x, \Phi) = x^3 - ax^2 + (a+1)x - 1$$

satisfy all conditions (11.1), . . . , (11.5) when

$2a + 3$ is prime (positive or negative) ,

$a \equiv 1, 2, 3 \pmod 5$ or $a \equiv 1, 2, 3, 6 \pmod 7$,

and

$$|a+3| > 3(|a|+2)^{1/3} .$$

Notice that $f(0) = -1$, $f(1) = 1$ and $f(2) = 9 - 2a$. Thus if $a \geq 5$, $f(x)$ has three positive roots. Each such Φ then would yield examples of closed orientable aspherical 7-dimensional solvmanifolds which admit no periodic maps.

In the other case with $\sigma_2 = \sigma_1 - 1$ another infinite family is obtained similarly from

$$f(x, \Phi) = x^3 - ax^2 + (a-1)x - 1$$

when $2\sigma_1 + 1 = 2a + 1$ is prime. For $a > 7$, $\int_0^1 f(x)dx > 0$, and hence $f(x, \Phi)$ has three positive roots since $f(0) = f(1) = -1$.

12. Appendix: Independent extensions

The group π of Section 3 was a very special central extension of K by Z^n. It is very likely possible and perhaps desirable to treat considerably more general extensions. To start, one must determine $\mathrm{Aut}(\pi)$ in terms of $\mathrm{Aut}(N)$ and K. This problem is examined here for a class of central extensions of Z^k by a discrete group N.

A central extension

$$0 \longrightarrow Z^k \longrightarrow \pi \longrightarrow N \longrightarrow 1$$

corresponds to an element in $H^2(N; Z^k)$. But $H^2(N; Z^k)$ is naturally isomorphic to the k-fold direct sum of $H^2(N; Z)$ with itself, and thus an element of $H^2(N; Z^k)$ can be regarded as an ordered k-tuple of elements, $A_1, \ldots, A_k$, in $H^2(N; Z)$.

(12.1) Definition: A central extension $0 \to Z^k \to \pi \to N \to 1$ is *independent* if and only if the corresponding $A_1, \ldots, A_k$ are linearly independent over $\mathbb{Z}$ in $H^2(N; Z)$.

Let us consider now an endomorphism $M: Z^k \to Z^k$. This induces a corresponding endomorphism $M_*: H^2(N; Z^k) \to H^2(N; Z^k)$. Under the isomorphism of $H^2(N; Z^k)$ with the k-fold direct sum of $H^2(N; Z)$ with itself this M_* is represented as follows. Write $M = [m_{i,j}]$ as a $k \times k$ integral matrix, then

$$M_*(A_1, \ldots, A_k) = \left(\sum_{j=1}^{k} m_{1,j} A_j, \ldots, \sum_{j=1}^{k} m_{k,j} A_j \right) .$$

(12.2) Lemma: *Suppose* $F \in H^2(N; Z^k)$ *is an independent extension and suppose* $M: Z^k \to Z^k$ *is an endomorphism for which* $M_*(F) = F$, *then* M *is the identity automorphism.*

Proof: Since $M_*(F) = F$ we must have

$$A_i = \sum_{j=1}^{k} m_{i,j} A_j$$

for all $1 \leq i \leq k$. But $A_1, \ldots, A_k$ are linearly independent over Z, thus $m_{i,j} = \delta_{i,j}$ and M is the identity matrix.

We shall apply this elementary remark as follows.

(12.3) Lemma: *Let* $0 \to Z^k \to \pi \to N \to 1$ *be an independent central extension for which the image of* Z^k *is a characteristic subgroup; then the kernel of* $\mathrm{Aut}(\pi) \to \mathrm{Aut}(N)$ *is naturally isomorphic to* $\mathrm{Hom}(N; Z^k)$.

Proof: We use $\alpha, \beta, \gamma \ldots$ for elements of Z^k and $u, v, w, \ldots$ for elements of N. Then $\pi = Z^k \times N$ with product

$$(\alpha, u) \cdot (\beta, v) = \left(\alpha + \beta + f(u, v),\ u \quad v\right)$$

where $f(u, v)$ is an extension cocycle. Suppose $\sigma \in \mathrm{Aut}(\pi)$ lies in the kernel of $\mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(N)$, then since Z^k is a characteristic subgroup there is a unique automorphism $M: Z^k \simeq Z^k$ such that $\sigma(\alpha, e) \equiv \big(M(\alpha), e\big)$. Furthermore, there is a function $h: N \longrightarrow Z^k$ such that $\sigma(0, u) \equiv \big(h(u), u\big)$. Since σ is an automorphism we have $\sigma(\alpha, u) = \big(M(\alpha) + h(u), u\big)$. However, from

$$\sigma\big((0, u) \cdot (0, v)\big) = \big(\sigma(0, u)\big)\big(\sigma(0, v)\big)$$

we also find that

$$M\big(f(u, v)\big) + h(u\,v) \equiv f(u, v) + h(u) + h(v),$$

or

$$M\big(f(u, v)\big) - f(u, v) \equiv h(v) - h(u\,v) + h(u)\,.$$

This last equation shows us that $M_*(F) = F$, where $F \in H^2(N; Z^k)$ is represented by $f(u, v)$. Since our extension was independent, however, it follows from (12. 2) that M is the identity and thus

$$h(v) - h(u\,v) + h(u) \equiv 0$$

So h is a homomorphism. Conversely, given a homomorphism $h: N \longrightarrow Z^k$ the corresponding element in the kernel of $\mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(N)$ is given by $\sigma_h(\alpha, u) = \big(\alpha + h(u), u\big)$.

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INJECTIVE OPERATIONS OF THE TORAL GROUPS II

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1. Introduction

In this note we shall give several applications of the fibering theorem of [3]. Let us recall that a toral action (T^k, M) is called *injective* if $f_*^x : \pi_1(T^k, 1) \longrightarrow \pi_1(M, x)$ is a *monomorphism*, where $f^x(t) = tx$. The principal geometric fact concerning injective actions is the *Splitting Theorem* [3; 3.1] :

If $(B_{\mathrm{im}(f_*^x)}, b_0)$ is the covering space of (M, x) associated to image (f_*^x), then there is a lifting of (T^k, M) to $(T^k, B_{\mathrm{im}(f_*^x)}) = (T^k, T^k \times W)$, where W is simply connected and the action is translation along the first factor. The covering transformations $N = \pi_1(M,x)/\mathrm{im}(f_*^x)$, operating on the right, commute with the action of T^k so that the following commutes:

$$\begin{tikzcd}
(T^k, T^k \times W, N) \arrow[r, "/T^k"] \arrow[d, "/N"] & (W, N) \arrow[d, "/N"] \\
(T^k, M) = \left(T^k, (T^k \times W)/N\right) \arrow[r, "/T^k"] & W/N = M/T^k .
\end{tikzcd}$$

When (T^k, M) has locally finite orbit structure then (W, N) is properly discontinuous. We may also start with a properly discontinuous action (W, N) and impose a left T^k — right N action on $T^k \times W$ compatible with the projections and actions on each factor. The collection of $(T^k - N)$ equivariant classes of $(T^k - N)$ actions are in 1:1 correspondence with the elements of $H^2(N; Z^k)$. Those N-actions that yield covering transformations on $T^k \times W$ correspond to the elements $\pi \in \mathcal{Q} \subset H^2(N; Z^k)$ on which the *characteristic homomorphism*

$$H^2(N; Z^k) \xrightarrow{i^*} H^2(N_x; Z^k) = \mathrm{Hom}(N_x, T^k)$$

yields an embedding $N_x \longrightarrow T^k$, for all isotropy groups N_x, $x \in W$. The coefficients are

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trivial N and N_x-modules and the set $\mathcal{A}$ of [2] then coincides with what we call the Bieberbach classes in [4]. If $\pi \in \mathcal{A} \subset H^2(N; Z^k)$, the central extension

$$0 \longrightarrow Z^k \longrightarrow \pi \longrightarrow N \longrightarrow 1$$

is, of course

$$0 \longrightarrow \mathrm{im}(f_*^x) \longrightarrow \pi_1(M,x) \longrightarrow \pi_1(M,x)/\mathrm{im}(f_*^x) \longrightarrow 1 .$$

If W is a contractible manifold then M is an aspherical manifold. (A manifold is called aspherical if it is $K(\pi, 1)$.) Then $a \in \mathcal{A}$ if and only if the central extension π is torsion free. Finally, any action of a connected Lie group on a closed aspherical (cohomology) manifold is necessarily an injective toral action.

Suppose $H_1(M,x)$ if finitely generated and let $f_*^x : H_1(T^k, 1) \longrightarrow H_1(M,x)$ be a monomorphism. Then the Fibering Theorem of [3] states:

> (T^k, M) is equivariantly homeomorphic to $(T^k, T^k \times_F Y)$ where F is finite. That is, M fibers equivariantly over $(T^k, T^k/F)$ with fiber Y and structure group the finite abelian subgroup F of T^k.

We also showed that (T^k, M) fibers equivariantly over $(T^k, T^k/F)$ for some F, if and only if the action (T^k, M) is injective and the element $\pi \in \mathcal{A} \subset H^2(N; Z^k)$ is a torsion element.

We shall show in the next section that knowledge of the rank of $H^2(N; Z)$ may allow us to restrict an injective action (T^k, M) to a suitable toral subgroup T^j so that (T^j, M) fibers equivariantly over T^j/F.

One of the features of the fibering theorem for closed smooth manifolds is that the smooth classification of (T^k, M^{n+k}), where f^x is a monomorphism on the first homology group, is reduced to the classification of smooth actions of finite abelian subgroups of T^k acting smoothly on closed n-manifolds. (The topological classification reduces to the topological classification of these groups on those closed cohomology n-manifolds which become locally Euclidean when producted with T^k.)

The possibility of this reduction for some injective (T^k, M^{n+k}) arising from a given (W^n, N), when W/N is compact, is equivalent to finding a normal subgroup N' of finite index in N acting freely in W and with abelian quotient. Unfortunately, as we shall see in § 3, this possibility does not always exist. The non-existence in general, § 4, is a consequence of an oriented bordism argument. Our proofs and results have been motivated by calculations

and examples we made earlier from the planar case, § 3. In §5 we develop the construction of §4 a little further.

2. Fibering Part of an Injective Action

Let (W, N) be a properly discontinuous action on a simply connected, semi-locally 1-connected, path connected, locally path connected, locally compact and paracompact space W. We shall also assume that N is finitely generated.

To each $\pi \in \mathfrak{a} \subset H^2(N, Z^k)$ there is associated a central extension

$$0 \longrightarrow Z^k \longrightarrow \pi \longrightarrow N \longrightarrow 1$$

and an injective action $\left(T^k, M(\pi)\right)$ whose splitting action $(T^k, T^k \times W, N)$ is represented by π. Of course $\pi_1\left(M(\pi), x\right)$ is the above extension of $Z^k = \mathrm{im}(f_*^x)$ by N.

2.1. Theorem: If $k > \mathrm{rank}\, H^2(N; Z)$ then for any $\pi \in \mathfrak{a} \subset H^2(N; Z^k)$ there is an integer $j \geq k - \mathrm{rk}\, H^2(N; Z)$ and a direct product decomposition $T^k = T^j \times T^{k-j}$ so that $(f^x | T^j)_* : H_1(T^j; Z) \longrightarrow H_1\left(M(\pi) : Z\right)$ is a monomorphism, and the image of $(f^x | T^{k-j})_*$ is a finite group. In particular, $M(\pi)$ fibers over T^j with finite abelian structure group.

Proof: Let us apply the Lyndon spectral sequence in homology to the central extension

$$0 \longrightarrow Z^k \longrightarrow \pi \longrightarrow N \longrightarrow 1$$

which is equivalent to

$$0 \longrightarrow \pi_1(T^k, e) \xrightarrow{f_*^x} \pi_1\left(M(\pi), x\right) \longrightarrow N \longrightarrow 1 .$$

Thus we have $\left\{E^r_{s,t}, d_r\right\} \Longrightarrow H_*(\pi; Z)$.

$$E^2_{s,t} \simeq H_s\left(N; H_t(Z^k; Z)\right) \simeq H_s(N; Z) \otimes H_t(Z^k; Z) .$$

In particular $d_2 : E^2_{2,0} \longrightarrow E^2_{0,1}$ yields, together with the edge homomorphism, an exact sequence

$$H_2(N; Z) \xrightarrow{d_2} H_1(Z^k; Z) \longrightarrow H_1(\pi; Z) .$$

The edge homomorphism is just $f_*^x : H_1(T^k; Z) \longrightarrow H_1(\pi; Z) = H_1(M(\pi), Z)$. Since $k > \operatorname{rk} H^2(N; Z) = \operatorname{rk} H_2(N; Z)$ we see $\operatorname{rank}(\operatorname{im}(f_*^x)) \geq k - \operatorname{rk} H^2(N; Z)$. Call the rank of this image j. Since $H_1(Z^k; Z) \cong Z^k$ is a free abelian group there is a direct sum decomposition

$$H_1(Z^k; Z) = A \oplus B$$

where $B \supset \operatorname{im}(d_2)$ and $\operatorname{rk}(B) = k - j$. Thus $f_*^x | A$ is a monomorphism and $f_*^x(B)$ is a finite group. Now corresponding to A and B there is a direct product decomposition $T^k = T^j \times T^{k-j}$ so that the image of $H_1(T^j; Z) \longrightarrow H_1(T^k; Z)$ is A while the image of $H_1(T^{k-j}; Z) \longrightarrow H_1(T^k; Z)$ is B.

We now may apply the fibration theorem to the induced action $(T^j, M(\pi))$ to see that $M(\pi)$ fibers over T^j with finite abelian structure group.

When W is 2-connected then $\operatorname{rk} H^2(N; Z) = \dim H^2(N; Q) = \dim H^2(W/N; Q)$. This often yields an easy way of computing $\operatorname{rk} H^2(N; Z)$.

2.2. Corollary: Let (T^k, M^{k+2}) be an effective action of the k-torus on a closed aspherical $(k+2)$-manifold. If $k > 1$, then M^{k+2} fibers over T^{k-1} with a finite abelian structure group.

Proof: The associated properly discontinuous action is (R^2, N) so that R^2/N is a closed 2-manifold. But $H^2(R^2/N; Q) \simeq H^2(N; Q)$ so that $\operatorname{rk} H^2(N; Z) \leq 1$, hence the corollary.

We remark that if M^{k+2} were not assumed closed but (T^k, M^{k+2}) was assumed injective then R^2/N is an open 2-manifold with finitely generated homology group. Then every element $a \in \mathcal{Q} \subset H^2(N; Z^k)$ has finite order and so (T^k, M^{k+2}) fibers over T^k with finite structure group and fiber an open 2-manifold.

We have seen that $(T^k, M(\pi))$ can be written as $(T^j \times T^{k-j}, T^j \times_F Y)$. This fibers equivariantly over $(T^j, T^j/F)$. Now $\operatorname{im} f_*^x = Z^j \times Z^{k-j}$ is a central subgroup of $\pi_1(M(\pi))$ and Z^j is mapped injectively into $\pi_1(T^j/F)$ with cokernel F. Furthermore, Z^{k-j} is mapped trivially into $\pi_1(T^j/F)$ by our previous discussion. Thus Z^{k-j} lies in the kernel, $\pi_1(T^j \times Y)$, of $\pi_1(M(\pi)) \longrightarrow F \longrightarrow 1$. Consequently, $(T^j \times T^{k-j}, M(\pi))$ lifts to $T^j \times Y$. The first factor T^j acts by left translation and the second factor T^{k-j} operates injectively on $T^j \times Y$ and hence injectively on Y.

Going back to the Corollary we see that the fiber must be a 3-dimensional closed Seifert manifold on which the circle group operates injectively and with finite structure group $F \subset T^{k-1}$.

In the particularly interesting case of closed aspherical 4-manifolds on which the 2-dimensional torus operates effectively, one may impose an analytic structure (via the splitting theorem), see [4]. These manifolds are aspherical elliptic surfaces with $Z \times Z$ in the center of their fundamental groups. Furthermore, any such aspherical elliptic surface is diffeomorphic to one of these and it can be shown that any two are diffeomorphic if and only if they have isomorphic fundamental groups. This is a consequence of the generalized Nielsen theorem and will be pursued elsewhere.

In another paper [5] the authors show that some of these closed elliptic surfaces have the property the fibering is actually trivial (the structure group must be enlarged) but not unique. The 4-manifold can be diffeomorphic both to $M_1^3 \times S^1$ and $M_2^3 \times S^1$ where M_i^3 are closed Seifert manifolds with different fundamental groups. For example, one may construct such a surface by taking the closed Seifert manifolds

$$M_1^3 = \{-1;(o,0,0,0);(5,1),(5,1),(5,1),(5,1),(5,1)\}$$

$$M_2^3 = \{-2;(o,0,0,0);(5,2),(5,2),(5,2),(5,2),(5,2)\}$$

with distinct fundamental groups. Their products with the circle are diffeomorphic (to an elliptic surface).

We would like to express our appreciation to Peter Orlik who pointed out to us that our fibering theorem implied aspherical closed 4-manifolds which admit effective T^2 actions fiber over the circle. He calculated this from an explicit presentation of the fundamental groups of these manifolds. The present 2.1 is inspired by his observation.

3. Planar Groups

In this section we shall examine $(\mathbb{R}^2, N)$ where N is a properly discontinuous effective action of a discrete group of orientation preserving homeomorphisms of the Euclidean plane with compact quotient $\mathbb{R}^2/N$.

It is known that

(i) every finite subgroup of N is cyclic,

(ii) every non-trivial finite subgroup has exactly one fixed point.

In fact, N may be presented as follows:

$$N = \left\{a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n \,\middle|\, [a_1, b_1]\cdots[a_g, b_g]q_1\cdots q_n,\ q_1^{\alpha_1}, \ldots, q_n^{\alpha_n}\right\},$$

with the conditions that $g \geq 0$, and if $g = 0$, $n \geq 3$, and if $n = 3$, $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 \leq 1$.

This is a presentation of an infinite crystallographic group or a Fuchsian group. Moreover the action $(\mathbb{R}^2, N)$ can be shown to be topologically equivalent to a holomorphic action.

Any finitely generated discrete group with a faithful matrix representation has a torsion free invariant subgroup N' with finite index by a theorem of A. Selberg. Of course the planar groups enjoy this property, although we shall not use this particular fact. However, we shall show by explicit computation that it is not possible, in general, to find an epimorphism onto a finite abelian group with torsionless kernel.

In [2; § 9] we obtained the following exact sequence:

3.1 $$0 \longrightarrow Z \xrightarrow{\beta} H^2(N; Z) \xrightarrow{\nu} Z_{\alpha_1} \oplus \ldots \oplus Z_{\alpha_r} \longrightarrow 0 .$$

The cyclic groups Z_{α_i} are isomorphic to N_{x_i}, where $\{x_i\}$ represent the different orbit types of $(\mathbb{R}^2, N)$ with finite stability groups.

3.2. Lemma: For each i, $1 \leq i \leq n$, the torsion subgroup of $H^2(N; Z)$ is mapped monomorphically onto a quotient quotient group of $Z_{\alpha_1} \oplus \ldots \oplus \hat{Z}_{\alpha_i} \oplus \ldots \oplus Z_{\alpha_n}$, where $\wedge$ denotes deletion. In particular, β maps a generator of Z onto a generator of the free part of $H^2(N; Z)$ if and only if $n = 0$.

Proof: We use the fact that

$$H^2(N; Z) = \mathrm{Hom}\left(H_2(N; Z), Z\right) \oplus \mathrm{Ext}\left(H_1(N; Z), Z\right)$$

and compute $H_1(N; Z)$.

$$H_1(N; Z) = \left\{ a_1 b_1, \ldots, a_g b_g, q_1, \ldots, q_n \;\middle|\; \begin{aligned} q_1 + q_2 + \ldots + q_n &= 0 \\ \alpha_1 q_1 \qquad\qquad &= 0 \\ \alpha_2 q_2 \qquad &= 0 \\ \alpha_n q_n &= 0 \end{aligned} \right\}$$

Thus, $H_1(N; Z) = \underbrace{Z \oplus \ldots \oplus Z}_{2g} \oplus \text{Torsion}.$

The torsion subgroup of $H^2(N; Z)$ is isomorphic to $\mathrm{Ext}\big(H_1(N; Z), Z\big)$ and therefore to a quotient group of $Z_{\alpha_1} \oplus \ldots \oplus \hat{Z}_{\alpha_i} \oplus \ldots \oplus Z_{\alpha_n}$, where Z_{α_i} is deleted for each i. This group is obtained by dividing out the relation $-\alpha_i(q_1 + \ldots + \hat{q}_i + \ldots + q_n) = 0$, for any i. This relation is always non-trivial if some α_j does not divide α_i, $i \neq j$.

3.3. Corollary: If some α_i is relatively prime to the remaining $\{\alpha_j\}$ then Torsion $H^2(N; Z)$ does not contain Z_{α_i}.

In fact, if all the α_i are relatively prime then $H^2(N; Z)$ is free cyclic. If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are all relatively prime to each other and the remaining α_j's, then Torsion $H^2(N; Z)$ is a quotient group of $Z_{\alpha_{k+1}} \oplus \ldots \oplus Z_{\alpha_n}$.

3.4. Theorem: Suppose there is a prime p which divides some α_i but no other α_j, then $H^2(N; Z)$ has no p-torsion. Furthermore, every epimorphism $\phi: N \longrightarrow F$, where $F \neq 1$ is finite, abelian, has p-torsion in its kernel. Consequently, no Bieberbach class in $H^2(N; Z^k)$ can have finite order.

Proof: The Corollary 3.3 implies that $H^2(N; Z)$ has no p-torsion and hence 3.2 implies that $H_1(N; Z)$ has no p-torsion. Let $Z_p \subset N_{x_i} \cong Z_{\alpha_i}$, and let K be the commutator subgroup. Take $g \in N_{x_i}$ so that order g is p. If g is not in K then gK must have order p in $H_1(N; Z)$ yielding a contradiction. Hence the kernel of ϕ, which must contain K, always has p-torsion. If $H^2(N; Z^k)$ were to contain a Bieberbach class of finite order for some k, then a normal $N' \subset N$ acting freely and with finite abelian quotient would exist. N' would have to be torsion free yielding a contradiction.

Here is an amusing consequence. In [4] it is shown that every Bieberbach class $a \in H^2(N; Z^{2k})$ gives rise to a family of holomorphic toral actions (T^{2k}, M^{2k+2}). The question as to whether the smooth closed M^{k+2} admits a structure of a projective algebraic variety is equivalent to whether or not a has finite order [4; §10 and 12]. Thus, if N is as in the theorem, the complex manifolds M^{2k+2} must not even admit a Kähler structure.

The situation is very different when the quotient space R^2/N is not compact. Turning again to 3.1 we see that $H^2(W/N; Z)$ is 0, and ν is bijective. Hence every $a \in H^2(N; Z^k)$ fibers equivariantly over T^k/F, by [3]. In particular all Bieberbach classes have finite order. Consequently, there exists a normal torsion free subgroup N' of N with finite cyclic

quotient, if the finite subgroups have bounded finite orders. Note here it is not necessary that N be finitely generated.

In the non-orientable case, the quotient $\mathbb{R}^2/N$ has either Z_2 or 0 for 2nd integral cohomology and once again each element of $H^2(N; Z^k)$ fibers over T^k and we may find a torsion free normal subgroup with finite cyclic quotient. Actually it appears that the most complicated algebraic structure for N occurs when $(\mathbb{R}^2, N)$ is orientation preserving and the quotient $\mathbb{R}^2/N$ is compact.

3.5. For any given properly discontinuous and effective (W, N) with W diffeomorphic to the Euclidean plane and W/N compact we may, with the aid of an element $a \in \mathcal{Q} \subset H^2(N; Z^k)$, construct a new (W, N_1) with N_1 satisfying the hypothesis of 3.4. Let us choose a point $x_0 \in W$ where N acts freely and form the injective T^k action $\left(T^k, (T^k \times W)/N\right) = (T^k, M^{n+k})$ determined by a. At a point $y \in M$ whose orbit is $\nu(x_0)$ we may take a disk slice S_y and remove a tubular neighborhood $S_y \times T^k$ of this orbit. Now sew in equivariantly $(T^k, T^k \times_{Z_p} S_y)$ by an equivariant homeomorphism of the boundaries. This introduces a stability group Z_p "at y". (The action of Z_p on the cell S_y we choose is just a linear action with exactly one fixed point at y, and we choose any embedding of Z_p in T^k.) This changes the space (T^k, M) to say (T^k, M_1). This new space and action is injective with splitting action $(T^k, T^k \times W, N_1)$. If we had chosen p a prime which does not divide the order of any stability group N_x, $x \in W$, then this new action (W, N_1) has p satisfying the hypothesis of 3.4 and (T^k, M_1) is injective. As a Bieberbach class it has infinite order. Furthermore, the fixed point set, $F(Z_p, M_1) \subset M_1$ is exactly one orbit of the T^k-action.

4. Examples of Injective S^1-actions

In this section we wish to construct a smooth, injective action (S^1, M^{2n+1}) on a closed oriented manifold, $n \geq 2$, which has exactly one non-trivial isotropy subgroup, $Z_p \subset S^1$, p an odd prime, and for which the fixed point set $F(Z_p) \subset M^{2n+1}$ is exactly one orbit of the S^1-action. For $n = 1$ it follows from the considerations of the previous section that such actions of S^1 on closed aspherical 3-manifolds can be constructed. Our initial thought was that with a minor adjustment to the procedure in section 3.5 we could produce examples for all values of n and p. This proved to be quite wrong and, as will be seen, we do not have a clear picture on the existence problem for such actions when $n \geq 2$. Since the action is injective we have $N = \pi_1(M)/\mathrm{im}(f_*^x)$ and we know (S^1, M) may be constructed from a Bieberbach class in $H^2(N; Z)$. It will follow, from a bordism argument, that for no value of $k > 0$ does $H^2(N; Z^k)$ contain a Bieberbach class of finite order.

Let us discuss our approach. Take an oriented lens space L^{2n-1} whose fundamental group is Z_p, p an odd prime. We ask if there is a compact oriented manifold B^{2n} for which $\partial B = L$ and for which the composite homomorphism

$$0 \longrightarrow H^2(\pi_1(B); Z) \longrightarrow H^2(B; Z) \longrightarrow H^2(L; Z)$$

is an epimorphism. Recall that $H^2(\pi_1(B); Z) \longrightarrow H^2(B; Z)$ is the edge homomorphism of the spectral sequence associated to the universal covering of B. We may interpret this situation as follows. With $x \in L = \partial B$ we obtain an isomorphic embedding $\pi_1(L, x) \longrightarrow \pi_1(B, x) = N$. Thus we have $Z_p \subset N$. Now there is a unitary action (Z_p, D) on the closed 2n-cell for which Z_p acts freely on ∂D and $\partial D/Z_p = L$. Thus we may form the properly discontinuous action $(D \times_{Z_p} N, N)$. On the other hand there is the universal covering action (B^*, N). With a suitable choice of orientation

$$\partial(B^*, N) = -(\partial D \times_{Z_p} N; N) = -\partial(D \times_{Z_p} N, N) \ .$$

Thus we have $W = B^* \cup (D \times_{Z_p} N)$, together with a properly discontinuous action (W, N). Since $n \geq 2$ it also follows that W is simply connected. Now there is a cohomology class $a \in H^2(N; Z)$ whose image under $H^2(N; Z) \longrightarrow H^2(B; Z) \longrightarrow H^2(L; Z) \simeq Z_p$ is a generator. Then clearly $a \in \mathfrak{a} \subset H^2(N; Z)$ is a Bieberbach class. The action (S^1, M^{2n+1}) associated to this Bieberbach class is a smooth injective action on a closed oriented manifold. The only isotropy subgroup is Z_p and its fixed point set is exactly one orbit. The quotient $W/N = M^{2n+1}/S^1$ is the union of B with the cone over $\partial B = L$.

(4.1) Theorem: There is no Bieberbach class of finite order in $H^2(N; Z^k)$, for any $k > 0$.

Proof: The oriented lens space L is canonically equipped with a map $f: L \rightarrow K(Z_p, 1)$ and thus an oriented bordism class $[L, f] \in \tilde{\Omega}^{SO}_{2n-1}(K(Z_p, 1))$ is defined. The class is not zero and its order was computed in [1, (36.1)]. Let $b \in H^1(L; Z_p)$ be the mod p cohomology class determined by f. Then the integral Bookstein $\delta(b) \in H^2(L; Z) \simeq Z_p$ is the generator.

Suppose now there is a Bieberbach class of finite order $a \in H^2(N; Z)$. Without loss of generality we can assume a has order p^r and that under $H^2(N; Z) \rightarrow H^2(L; Z)$ the image of a is $\delta(b)$. Thus we can choose $c \in H^1(N; Z_{p^r})$ with $\delta(c) = a$. Then under $H^1(N; Z_{p^r}) \longrightarrow H^1(L; Z_{p^r})$ the image of c agrees with the image of b under $H^1(L; Z_p) \simeq H^1(L; Z_{p^r})$. This

implies that the map $g: B \longrightarrow K(Z_{p^r}, 1)$ corresponding to $c \in H^1(B; Z_{p^r})$ is an extension of the composition $L \xrightarrow{f} K(Z_p, 1) \longrightarrow K(Z_{p^r}, 1)$. That is, $[L, f]$ lies in the kernel of $\tilde{\Omega}^{SO}_{2n-1}(K(Z_p, 1)) \longrightarrow \tilde{\Omega}^{SO}_{2n-1} K(Z_{p^r}, 1)$. However, in [1, (37.2)] it was shown that the bordism homomorphism induced by $K(Z_p, 1) \longrightarrow K(Z_{p^r}, 1)$ is a monomorphism. This contradiction then proves the theorem for $k = 1$. Since up to conjugacy there is only the one isotropy subgroup in (W, N) it also follows that $H^2(N; Z^k)$ contains no Bieberbach class of finite order.

Obviously we must still produce the manifold B with the required properties. Suppose $Z_p \subset G$ is an embedding of Z_p into some (discrete) group such that $H^*(G; Z) \longrightarrow H^*(Z_p; Z)$ is an epimorphism. There is induced a map $K(Z_p, 1) \longrightarrow K(G, 1)$ and an Ω^{SO}_*-module homomorphism $\tilde{\Omega}^{SO}_*(K(Z_p, 1)) \longrightarrow \tilde{\Omega}^{SO}_*(K(G, 1))$. If $[L, f]$ lies in the kernel of this homomorphism, then there will exist a compact oriented B with $\partial B = L$ and a map $F: B \longrightarrow K(G, 1)$ which extends the composition of f with $K(Z_p, 1) \longrightarrow K(G, 1)$. But then we have

$$\begin{array}{ccccc}
H^2(G; Z) & \longrightarrow & H^2(\pi_1(B); Z) & \longrightarrow & H^2(B; Z) \\
\downarrow & & & & \downarrow \\
H^2(\pi_1(L); Z) & & \longrightarrow & & H^2(L, Z)
\end{array}$$

and since the first vertical arrow is an epimorphism, so is the second.

So that problem becomes that of finding some suitable choices for G. We shall give some possibilities, but these are rather ad hoc. For each $k \geq 1$ we regard $H^*(Z^{2k}; Z)$ as an exterior algebra on 1-dimensional generators $e_1, \ldots, e_{2k}$. Let $\sum_{i<j} e_i \wedge e_j = c \in H^2(Z^{2k}; Z)$. With p a fixed odd prime we introduce the central group extension

$$0 \longrightarrow Z_p \longrightarrow G(k) \longrightarrow Z^{2k} \longrightarrow 0$$

given by the mod p reduction of c into $H^2(Z^{2k}; Z_p)$.

4.2. <u>Definition</u>: For each integer $k \geq 1$ a set $P(k)$ of odd primes is defined by $p \in P(k)$ if and only if

$$\cup c^{k-i}: H^i(Z^{2k}; Z_p) \simeq H^{2k-i}(Z^{2k}; Z_p)$$

for all $0 \leq i \leq k$.

We wish to prove then

4.3. Lemma: If $p \in P(k)$ then $H^*(G(k); Z) \longrightarrow H^*(Z_p; Z)$ is an epimorphism and $H_i(G(k); Z)$ contains no p-torsion for $0 \leq i \leq k$.

The group $G(k)$ can also be constructed as follows. Let $x \in H^2(CP(\infty); Z)$ generate $Z[x] = H^*(CP(\infty); Z)$. Take the principal S^1-bundle $\mu: K \longrightarrow T^{2k} \times CP(\infty)$ with characteristic class $c \otimes 1 - 1 \otimes px$. We assert that $K = K(G(k), 1)$. Indeed, the composition

$$K \xrightarrow{\mu} T^{2k} \times CP(\infty) \longrightarrow T^{2k}$$

is a fiber space over T^{2k} with fiber $K(Z_p, 1)$. We merely observe that the principal S^1-bundle over $\{pt\} \times CP(\infty) \subset T^{2k} \times CP(\infty)$ induced by the inclusion from $\mu: K \longrightarrow T^{2k} \times CP(\infty)$ has characteristic class $-px$, and thus $K(Z_p, 1)$ is the total space of this induced bundle. Furthermore, if $\mu^*(x) = a \in H^2(K; Z)$ then under the homomorphism $i^*: H^2(K; Z) \longrightarrow H^2(K(Z_p, 1); Z)$, which is induced by the inclusion of a fiber, the image $i^*(a)$ generates $H^*(Z_p; Z)$.

We may now consider the Gysin sequence of $\mu: K \longrightarrow T^{2k} \times CP(\infty)$. We immediately observe, however, that cupping with the characteristic class $c \otimes 1 - 1 \otimes px$ in $H^*(T^{2k} \times CP(\infty); Z) \simeq E(e_1, \ldots, e_{2k}) \mathbin{\hat{\otimes}} Z[x]$ is a monomorphism. Thus the Gysin sequence collapses and $\mu^*: H^*(T^{2k} \times CP(\infty); Z) \longrightarrow H^*(K; Z) \longrightarrow 0$ is an epimorphism. We may therefore write

$$H^*(G(k); Z) = E(e_1, \ldots, e_{2k}) \mathbin{\hat{\otimes}} Z[x]/(c \otimes 1 - 1 \otimes px) .$$

We may also consider the mod p Gysin sequence associated with $\mu: K \longrightarrow T^{2k} \times CP(\infty)$. But modulo p, the characteristic class is just $c \otimes 1$. Recall that since $p \in P(k)$, $\cup c^{k-i}: H^i(T^{2k}; Z_p) \simeq H^{2k-i}(T^{2k}; Z_p)$, $0 \leq i \leq k$ so that surely $\cup c: H^i(T^{2k}; Z_p) \longrightarrow H^{i+2}(T^{2k}; Z_p)$ is a monomorphism if $0 \leq i < k$. From the mod p Gysin sequence, then we conclude that $\mu^*: H^2(T^{2k} \times CP(\infty); Z_p) \longrightarrow H^i(K; Z_p)$ is an epimorphism if $0 \leq i \leq k+1$. Obviously we then have $H^i(G(k); Z) \longrightarrow H^i(G(k); Z_p) \longrightarrow 0$ an epimorphism for all $0 \leq i \leq k+1$ also. Translating this into homology we have the proof of 4.2.

To put this into a statement about $\tilde{\Omega}_*^{SO}(K(G(k), 1))$ we localize the oriented bordism functor at the odd prime p. That is, we use $\Omega_*^{SO}(\cdot) \otimes_Z Q_{(p)}$, where $Q_{(p)}$ is the ring of rational numbers with denominator prime to p. Associated with this localized homology

theory is the spectral sequence $\left\{E^r_{p,q}, \partial_r\right\} \Longrightarrow \Omega^{SO}_*(\cdot) \otimes Q_{(p)}$ with $E^2_{p,q} \simeq H_p(\cdot, Q_{(p)}) \otimes \Omega^{SO}_q$. Since $H_i\big(K(G(k),1), Q_{(p)}\big)$ is torsion free for $0 \leq i \leq k$, it follows $\tilde{\Omega}^{SO}_i\big(K(G(k),1)\big) \otimes Q_{(p)}$ is also torsion free for $0 \leq i \leq k$. Hence we have

4.4. Lemma: The reduced oriented bordism groups $\tilde{\Omega}^{SO}_i\big(K(G(k),1)\big)$ contain no p-torsion if $0 \leq i \leq k$.

On the other hand, $\tilde{\Omega}^{SO}_*(K(Z_p, 1))$ consists entirely of p-torsion, so if $p \in P(k)$ and $2n-1 \leq k$ then our element $[L^{2n-1}_p, f]$ lies in the kernel of $\tilde{\Omega}^{SO}_{2n-1}(K(Z_p, 1)) \longrightarrow \tilde{\Omega}^{SO}_{2n-1}\big(K(G(k), 1)\big)$.

Since $\cup c^{n-i} : H^i(Z^{2k}; Q) \simeq H^{2k-i}(Z^{2k}; Q)$ for all $0 \leq i \leq k$ it follows that each set $P(k)$ contains almost all primes. However, if $p \in P(k)$ then $p > k$. So, for a fixed prime, we are restricted to lens spaces of dimension $2n-1 \leq k < p$. We would like to know some alternatives to the groups $G(k)$ used here.

5. An Application of 4.1.

5.1. The purpose of this section is to show how one may begin with a smooth toral action on a manifold and alter the action and the manifold by a controlled introduction of new orbit types. The alteration will be done by removing the interior of an invariant tubular neighborhood of a principal orbit and sewing in, equivariantly, a manifold whose toral action is free except for one orbit where the stability group is precisely Z_p. The manifolds and the properly discontinuous actions (W, N) constructed in section 4 will form our basic building blocks. The control we desire is that when we begin with an injective action we must end with an injective action. The action is to remain unchanged on the original manifold away from tubular neighborhood of the chosen principal orbit. We also wish to be able to construct the new splitting action from the old one and our basic building blocks. Finally, the end result of the construction should eliminate all Bieberbach classes of finite order.

5.2. The construction is straightforward. Let $(T^\ell, M_1^{2n+\ell})$ be a smooth effective toral action on $M_1^{2n+\ell}$. Let $x \in M_1^{2n+\ell}$ lie on a principal orbit. Choose a smooth $2n$-cell slice D^{2n}_x at x. We obtain an invariant tubular neighborhood

$$(T^\ell, T^\ell \times D^{2n}_x) .$$

The orbit map, $\nu_1 : M_1 \longrightarrow M_1/T^\ell = V_1$, restricted to $T^\ell \times D_x^{2n}$ is just a principal bundle map equivalent to projection onto the second factor.

In section 4, we constructed for each prime $p \in P(k)$, a smooth simply connected 2n-dimensional manifold W and an effective properly discontinuous group N of orientation preserving diffeomorphisms of W with compact quotient $W/N = V$. The action is free everywhere except for a single orbit at which the stability group is isomorphic to Z_p. The orbit space V is the cone over a lens space attached to the boundary ∂B of the manifold B found in section 4. The vertex of the cone is the single orbit with non-trivial isotropy subgroup.

Recall that by choosing a Bieberbach class in $H^2(N; Z)$ we found an orientable manifold M^{2n+1} with an effective injective S^1-action. $(S^1, M^{2n+1}) = \left(S^1, (S^1 \times W)/N\right)$ and so the action is free everywhere except for a single orbit over the vertex of the cone in the orbit space $M^{2n+1}/S^1 = W/N = V$. Clearly, we could have as well chosen an element of $H^2(N; Z^\ell)$ with a similar property; for example, $M^{2n+\ell} = T^{\ell-1} \times M^{2n+1}$, with the obvious T^ℓ-action. Let us choose $y \in M^{2n+\ell}$ on a principal orbit and remove the interior of an invariant tubular neighborhood $(T^\ell, T^\ell \times D_y)$ as above. The boundaries of the deleted orbit spaces $V_1 - D_x$ and $V - D_y$ are $(2n-1)$-spheres. The actions over these boundaries are smooth product actions and by choosing cross-sections we may attach $M_1 - (T^\ell \times D_x^{2n})$ equivariantly to $M - (T^\ell \times D_y^{2k})$. We let

$$h : S^{2n-1} \longrightarrow \partial(V - D_y^{2n})$$

$$h_1 : S^{2n-1} \longrightarrow \partial(V_1 - D_x^{2n})$$

be orientation preserving and orientation reversing diffeomorphisms respectively and attach by $h_1^{-1} \circ h$. We use the cross-sections to ν and ν_1 to equivariantly attach $M - (T^\ell \times D_y^{2n})$ to $M_1 - (T^\ell \times D_x^{2n})$. There results a new (oriented, if $M^{2n+\ell}$ was assumed oriented) manifold $'M^{2n+\ell}$ with an effective T^ℓ toral action. The orbit space is $'V = (V - D_y^{2n}) \cup (V_1 - D_x^{2n})$ and the action $(T^\ell, {'M})$ agrees with $\left(T^\ell, M_1 - (T^\ell \times D_x^{2n})\right)$ on the restriction to $M_1 - (T^\ell \times D_x^{2n})$. Only one orbit type different from principal orbits is introduced. Of course it is the orbit which lies over the vertex of the cone over L in $V - D_y^{2n} \subseteq {'V}$.

5.3. Let us now suppose $(T^\ell, M_1^{2n+\ell})$ is a smooth injective action. From 4.1, $(T^\ell, M^{2n+\ell})$ is also injective. Several applications of Van Kampen's theorem yield that the new action $(T^\ell, {'M}^{2n+\ell})$ is also injective, since $n \geq 2$.

We are interested in determining the new properly discontinuous action ('W, 'N) arising from the splitting diagram:

$$\begin{tikzcd}
(T^{\ell}, T^{\ell} \times 'W, 'N) \arrow[r, "/T^{\ell}"] \arrow[d, "/'N"] & ('W, 'N) \arrow[d, "/'N"] \\
(T^{\ell}, 'M) \arrow[r, "/T^{\ell}"] & 'M/T^{\ell}
\end{tikzcd}$$

Let (W,N) and (W_1,N_1) denote the properly discontinuous actions arising from the splitting actions for $(T^{\ell}, M^{2n+\ell})$ and $(T^{\ell}, M_1^{2n+\ell})$.

5.4. Proposition: The group $'N$ is isomorphic to the free product $N * N_1$. For the action $('W, 'N)$, all Bieberbach classes in $H^2('N; Z^k)$, for any $k \geq 1$, are of infinite order.

Proof: In 5.2 we have already described the sewing that must be done on the deleted orbit spaces $V-D_y^{2n}$ and $V_1-D_x^{2n}$. We shall consider a number of disjoint copies of $W-\nu^{-1}(D_y^{2n}) = U$ and $W_1-\nu_1^{-1}(D_x^{2n}) = U_1$. The boundary of U and U_1 consists of a number of copies of the sphere S^{2n-1} one for each element of N and N_1 respectively. Thinking of U and U_1 as simply connected branched covering spaces of $V-D_y^{2n}$ and $V_1-D_x^{2n}$ we choose a base point v in the sphere S^{2n-1} and then base points $h(v)$ and $h_1(v)$ in the boundaries of $V-D_y^{2n}$ and $V_1-D_x^{2n}$. We may then choose u and u_1 in U and U_1 so that $\nu(u) = h(v)$ and $\nu_1(u_1) = h_1(v)$ and so that u and u_1 correspond to the constant paths issuing from $h(v)$ and $h_1(v)$ when U and U_1 are thought of as branched covering spaces. We lift the diffeomorphisms h and h_1 to the boundary component of U and U_1 containing u and u_1 respectively. Thus we attach our first copies of U to U_1.

Let $\gamma = \alpha_1^{r_1}\beta_1^{s_1}\ldots\alpha_m^{r_m}\beta_m^{s_m}$ be an arbitrary word in $N * N_1$, with $\alpha_i^{r_i} \in N$ and $\beta_i^{s_i} \in N_1$. The sphere $S^{2n-1} \subset U$ containing u is transformed by the automorphism $\alpha_1^{r_1}: U \longrightarrow U$ into the sphere containing $u\alpha_1^{r_1}$. We attach a new copy of U_1 to U at $S^{2n-1}\cdot\alpha_1^{r_1}$ by means of $\alpha_1^{r_1} \circ h = h_1$. Now on the copy of U_1, attached to U at $S^{2n-1}\cdot\alpha_1^{r_1}$, we have the automorphism induced by $\beta_1^{s_1}$. This sends $S^{2n-1}\cdot\alpha_1^{r_1}$ to a boundary component $S^{2n-1}\cdot\alpha_1^{r_1}\beta_1^{s_1}$ of the attached copy of U_1. In terms of the original copy of U_1 this coincides with $S^{2n-1}\cdot\beta_1^{s_1}$. We now attach another copy of U to $(U \cup U_1)$ along $S^{2n-1}\cdot\alpha_1^{r_1}\beta_1^{s_1}$ by $\alpha_1^{r_1} \circ h(w) = \beta_1^{s_1} \circ h_1(w)$, $w \in S^{2n-1}$. We may obviously continue this construction for each word $\gamma \in N * N_1 = {'N}$. Any point in $'W$ may be thought of as a point in either of the original

copies of U or U_1 transformed by a word on $'N$. The action $('W, 'N)$ is now clear: If $'w = u\gamma$, where $u \in U$ or U_1, then $'w\gamma_1 = u\gamma\gamma_1$. We have left out the details that two equivalent words give rise to the same attachings and the same transformation. It should also be clear that the construction given describes the properly discontinuous action associated to the splitting action for $(T^\ell, 'M^{2n+\ell})$.

It only remains to show that no Bieberbach class in $H^2('N; Z^k)$ is of finite order. Since $H^2('N; Z^k) \cong H^2(N; Z^k) \oplus H^2(N_1; Z^k)$, any Bieberbach class $b \in H^2('N; Z^k)$ will necessarily have to restrict to Bieberbach classes for the actions (U, N) and (U_1, N_1). But by 4.1 $H^2(N; Z^k)$, for (W, N) and hence also for (U, N), has no Bieberbach classes of finite order.

One novel feature of the preceding is that we do not need to require that $'W/'N$ be compact to obtain no Bieberbach classes of finite order. Thus it is impossible to find a homomorphism of $N * N_1$ onto a finite abelian group with a kernel K which acts freely on $'W$ whenever we choose a prime $p \in P(k)$. Furthermore, the construction may be iterated any number of times by either "adding on building blocks" of the form arising from 4.1 or just by combining in the manner described a different toral action with $(T^\ell, M^{2n+\ell})$.

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HOLOMORPHIC SEIFERT FIBERING

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1. Introduction

In our previous paper, [13, (8.1)], we considered the construction of a toral group acting effectively as a group of diffeomorphisms on a closed aspherical manifold. We began with a properly discontinuous group of diffeomorphisms on a contractible manifold, (W,N), for which the quotient space $W/N = V$ is compact; then to each torsionless central extension $0 \to Z^k \to \pi \to N \to 1$ we associated a smooth action (T, M) of a k-torus on a closed aspherical manifold with fundamental group π and $M/T = V$. It was shown that every effective toral group action on a closed aspherical manifold can be constructed in this way. We might now ask about the geometric significance of a torsionless, but non-central extension of Z^k by N. Thus we fix a homomorphism $\phi: N \to GL(k, Z)$ and consider it to be an abstract kernel. Then to each torsionless group extension realizing this kernel, $0 \to Z^k \to \pi \to N \to 1$ we associate a smooth Seifert fibering of a closed aspherical manifold with fundamental group π

$$\begin{tikzcd} & M \arrow[d, "\iota"] \\ W \arrow[r, "\nu"] & V \end{tikzcd}$$

The generic fibre is the k-torus T. Each singular fibre $\iota^{-1}(\nu(w))$ is a locally flat Riemannian manifold whose fundamental group is the induced extension $0 \to Z^k \to \pi_w \to N_w \to 1$, where N_w is the (non-trivial) isotropy subgroup of W.

We are attempting in this paper to generalize in two directions simultaneously. We have just indicated the first direction. We are also interested in the construction of a holomorphic action of a complex toral group on a complex analytic manifold. Such an action can have at most finite isotropy groups. In this paper we often assume $T \times M \to M$ is holomorphic. For this type of action, H. Holmann, in [20], has demonstrated the existence of

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holomorphic slices. Thus, with $\dim_c T = k$, $\dim_c M = n$, for each $x \in M$ there is a complex linear representation on C^{n-k} of the isotropy subgroup T_x and an open invariant neighborhood of the origin $0 \in K \subset C^{n-k}$ such that there is an equivariant holomorphic equivalence between $(T, T \times_{T_x} K)$ and an open invariant neighborhood of the orbit of x taking $((e, 0))$ to x. By $T \times_{T_x} K$ we mean the quotient of $T \times K$ with respect to the action of T_x given by $\sigma(t, v) = (t\sigma^{-1}, \sigma v)$ for all $\sigma \in T_x$.

We shall construct such actions from a properly discontinuous group of holomorphic transformations (W, N) on a complex analytic manifold. To accomplish this we introduce a short exact sequence of sheaves

$$0 \to \mathfrak{Z}^{2k} \to \mathcal{O}(W)^k \to \mathfrak{J} \to 0$$

over W. On each sheaf is a left action of N as a group of sheaf cohomomorphisms so that we are presented with a short exact sequence of sheaves with operators. We use $\mathfrak{Z}^{2k} \to W$ to denote the constant sheaf $Z^{2k} \times W \to W$ together with the action of N. By $\mathcal{O}(W)^k \to W$ we denote the sheaf of germs of holomorphic maps into C^k and by $\mathfrak{J} \to W$ the sheaf of germs of holomorphic maps into the complex toral group T.

In section 2 we introduce the cohomology of the group N with coefficients in a sheaf with operators $\mathcal{S} \to W$. We denote these by $H^*(N; \mathcal{S})$. These groups were introduced by Kodaira in [23, sec. 13] as playing a role in his construction of analytic surfaces. We give an expanded treatment of this topic, and by the end of section 2 we have at least determined $H^*(N; \mathcal{O}(W)^k)$. In section 3 we take up $H^*(N; \mathfrak{Z}^{2k})$, and especially $H^2(N; \mathfrak{Z}^{2k})$. Although there are formidable obstacles to a complete determination, it is possible by a combination of algebraic and topological considerations, to study this group. (The spectral sequences, introduced in § 2 (cf. also [12]) associated with $H^*(N; \mathcal{S})$ are due to H. Cartan and J. Leray. The groups, themselves, are a more abstract setting of the cohomology associated with a covering and topologically can be realized as the cohomology of the Borel space associated with an action. A. Grothendiek introduced these groups in [18, chap. V] and studied the spectral sequences associated with them. Although Grothendiek's definition of $H^*(N; \mathcal{S})$ is not the same as the one adopted here, which is modelled after Kodaira's, it is easily seen to be equivalent.) We introduce the subset $\mathcal{B} \subset H^2(N; \mathfrak{Z}^{2k})$ of Bieberbach classes. This is the appropriate generalization of the torsionless group extensions as they were used in [13, sec. 8].

In section 4 we make use of $\longrightarrow H^1\left(N;\mathcal{O}(W)^k\right) \xrightarrow{e_*} H^1(N;\mathcal{J}) \xrightarrow{\delta} H^2(N;\mathfrak{Z}^{2k}) \xrightarrow{i_*} H^2\left(N;\mathcal{O}(W)^k\right) \longrightarrow \dots$. To each $\tau \in H^1(N;\mathcal{J})$ we associate a holomorphic left principal T-bundle with operators $(T, B_\tau, N) \longrightarrow (W, N)$. The group N acts freely on B_τ as a properly discontinuous group of covering transformations if and only if $\delta(\tau) \in H^2(N;\mathfrak{Z}^{2k})$ is a Bieberbach class. In such cases we put $M_\tau = B_\tau/N$ and when Φ is trivial T acts on M_τ as a holomorphic transformation group with $M_\tau/T = V$. Rather obviously M_τ is closed if and only if V is compact and M_τ is aspherical if and only if W is. We may replace τ by $\tau + e_*(v)$. Then $\delta\left(\tau + e_*(v)\right) = \delta(\tau)$ is still a Bieberbach class and we regard $(T, M_{\tau+e_*(v)})$ as an equivariant deformation of (T, M_τ). If $a \in \mathcal{B} \subset H^2(N;\mathfrak{Z}^{2k})$ is a Bieberbach class then it has a holomorphic realization if and only if it lies in the kernel of $H^2(N;\mathfrak{Z}^{2k}) \longrightarrow H^2\left(N;\mathcal{O}(W)^k\right)$. Every Bieberbach class of finite order has a holomorphic realization.

Our interest in the holomorphic case was developed from a desire to find examples of closed aspherical complex manifolds. At the same time we realize that there is an interest in holomorphic actions in general. We hope that by discussion of this method of construction and treating it from the viewpoint of Kodaira's groups $H^*(N;\mathcal{S})$ we might stimulate a little further interest. In (7.2) we characterize those holomorphic actions which can be constructed by the present device. In sections 8 and 10 we study a holomorphic analogue of the fibration results of [14]. Sections 7 and 12 describe the changes needed for the smooth and continuous cases while section 13 discusses replacing the toral group by a non-compact abelian Lie group. Other applications of the fibering theorems are also treated in section 12.

Much of our work was accomplished without knowledge of Holmann's results, [20] and [21]. In fact, we originally found it necessary to assume the existence of local holomorphic slices. P. Wagreich called our attention to Holmann's theorem on slices as well as his paper on Seifert-fibre spaces. Of course, our definition of Seifert fibering is also a fibering in the sense of Holmann, however the respective approaches are different and almost complementary. What we may have lost in utmost generality is compensated by our gain in effective computability. Since our paper is partly expository, we have tried to make certain aspects of it self-contained. Our need for references to cohomology of complex manifolds is quite modest and would be met by section 15, p. 114 of [19]. The reader does not need to know the contents of [33] and [13], although our title indicates this paper is a generalization of the structure in [33] while the construction introduced in [13] is pushed to its limit.

There is no shortage of special examples. A completely comprehensive treatment is presently available for $\dim_R W = 2$, which is treated at various points in the text. In section 9 we consider a restricted, but not uncommon, class of examples with $\dim_R W = 4$, using the commutator subgroup, $[N, N]$, to detect Bieberbach classes. We also wish to point out

that a fairly broad class of mathematical objects falls within the province of Seifert fiberings. For example, by choosing W to be a point and N a finite group, with Φ non-trivial, the construction of Seifert fiberings over a point leads to the class of flat manifolds. Or, if W is complex projective n-space, CP_n, and N has a free unitary representation in $U(n+1)$, the smooth construction yields the manifolds of constant positive curvature. Classification of these structures often can be reduced to an appropriate classification of Seifert fiberings. We illustrate this in §7, 9, 11, 12 and 13. As an illustration at the end of §12 we point out how, in particular, one may topologically classify, by their fundamental groups, almost all of the elliptic surfaces of Kodaira having only multiple fibers. These applications are another aspect of Seifert fiberings which we hope will stimulate further investigations.

Finally, we wish to thank T. Suwa for several very helpful conversations.

2. Sheaves with Operators

A complex torus is determined by the choice of a real basis for C^k so that to each complex torus there is associated a short exact sequence

$$0 \to Z^{2k} \to C^k \to T \to 0 .$$

By $\mathrm{Aut}(T) \subset GL(k, C)$ we denote the subgroup of elements which preserve the image of $Z^{2k} \to C^k$. An element of $\mathrm{Aut}(T)$ may be regarded as an automorphism of any one of the three groups Z^{2k}, C^k or T.

Let (W, N) denote a properly discontinuous group of holomorphic transformations on a complex analytic manifold W with quotient the analytic space $V = W/N$, [9, sec. 4]. Let us fix a homomorphism $\Phi: N \to \mathrm{Aut}(T)$. We shall write $\alpha_* = \Phi(\alpha)$. The symbol S will denote any one of the three groups in the short exact sequence. To each open set $U \subset W$ we associate the abelian group $\mathrm{map}(U, S)$ of all holomorphic maps of U into S. In case $S = Z^{2k}$, this is simply the abelian group of all functions on U into Z^{2k} which are constant on the components of U. In each case we receive a sheaf of germs $\mathcal{S} \to W$. There is a natural left action of N on $\mathcal{S}$ as a group of sheaf cohomomorphisms, [12]. To each open $U \subset W$ and each $\alpha \in N$ we associate

$$\mathrm{map}(U\alpha, S) \xrightarrow{\alpha^U} \mathrm{map}(U, S)$$

where $\alpha^U(f)(w) = \alpha_*\big(f(w\alpha)\big)$. Thus $\mathcal{S} \to W$ becomes a sheaf with operators N.

We shall be concerned with $H^*(N; \mathcal{S})$, the cohomology of the group N with coefficients in this sheaf with operators. This will be the cohomology of a double complex arising from a differential sheaf, [6, p. 128], on the quotient space V. Let $\{\beta_j(N), \partial_j\}$ denote the un-normalized bar resolution of Z by free left $Z(N)$-modules, [25, p. 114]. For each open set $U \subset V$, $\mathrm{map}\big(\nu^{-1}(V), S\big)$ is a left $Z(N)$-module with

$$\alpha_{\#}(f)(w) = \alpha_*\big(f(w\alpha)\big) .$$

Thus we may introduce a differential sheaf

$$\mathcal{L} : \mathcal{L}^0 \to \mathcal{L}^1 \to \dots \to \mathcal{L}^j \to$$

over V by assigning to each open $U \subset V$ the abelian group

$$\mathrm{Hom}_{Z(N)}\Big(\beta_j(N), \mathrm{map}\big(\nu^{-1}(U), S\big)\Big) .$$

There is then the double complex $L^{*,*} = \sum L^{i,j}$, where $L^{i,j} = C^i(V; \mathcal{L}^j)$, which is

equipped with two coboundary operators $d': L^{i,j} \to L^{i+1,j}$ and $d'': L^{i,j} \to L^{i,j+1}$ satisfying $d'd''+d''d' = 0$. We define $H^*(N;\mathcal{S})$ to be the cohomology of $L^{*,*}$ with respect to $d = d'+d''$.

The first spectral sequence [6, p. 128], is described in terms of sheaves $h^j \to V$ which are defined by assigning to each open $U \subset V$ the cohomology group $H^j\left(N; \mathrm{map}\left(\nu^{-1}(U), S\right)\right)$. Thus we have a spectral sequence $\left\{{}'E_r^{i,j}, dr\right\} \Rightarrow H^*(N;\mathcal{S})$ with

$$'E_2^{i,j} \simeq H^i(V; h^j) \quad .$$

Let $\mathcal{S}^* \to V$ be the direct image of $\mathcal{S} \to W$ under the quotient map $\nu: W \to V$. Applying the Leray spectral sequence of a map to ν [6, p. 140], we immediately conclude that $H^*(W;\mathcal{S}) \simeq H^*(V;\mathcal{S}^*)$. Both of these cohomology groups have a natural $Z(N)$-module structure and the isomorphism is a $Z(N)$-module isomorphism.

Now with j fixed the cohomology of $\left\{C^i(V,\mathcal{L}^j), d_i'\right\}$ is $\mathrm{Hom}_{Z(N)}\left(\beta_j(N), H^i(V;\mathcal{S}^*)\right)$ $\simeq \mathrm{Hom}_{Z(N)}\left(\beta_j(N), H^*(W;\mathcal{S})\right)$. Hence we may also say that there is a second spectral sequence $\left\{{}''E_r^{i,j}, d_r\right\} \Rightarrow H^*(N;\mathcal{S})$ for which

$$''E_2^{i,j} \simeq H^i\left(N; H^j(W;\mathcal{S})\right) \quad .$$

Let us proceed now to determine the stalks of the sheaves $h^j \to V$. To do this we shall study the $Z(N)$-module on each stalk of the direct image sheaf $\mathcal{S}^* \to V$. We select a point $w_0 \in W$. Since (W, N) is properly discontinuous, the isotropy subgroup of w_0, denoted for now by $H \subset N$, is finite. Furthermore, there are arbitrarily small open neighborhoods, K, of w_0 for which

$$K \cdot H = K$$

$$K \cap K\alpha \neq \emptyset \text{ if and only if } \alpha \in H.$$

For each such K we define a $Z(H)$-module structure on $\mathrm{map}(K, S)$ by

$$h_*(g)(w) = h_*\left(g(wh)\right) \quad .$$

In this way the stalk $\mathcal{S}_{w_0}$ becomes a $Z(H)$-module. We introduce $\mathrm{Hom}_{Z(H)}\left(Z(N), \mathcal{S}_{w_0}\right)$ on which the left $Z(N)$-module structure is defined as follows. Let $\zeta: Z(N) \to \mathcal{S}_{w_0}$ be a $Z(H)$-module homomorphism. Then, for $\alpha \in N$, let

$$\alpha_\#(\zeta)(\beta) = \zeta(\beta\alpha) \quad .$$

(2.1) Lemma: If $\mathcal{S}^*_{\nu(w_0)}$ is the stalk of $\mathcal{S}^* \to V$ at $\nu(w_0)$ then there is a canonical $Z(N)$-module isomorphism

$$\mathcal{S}^*_{\nu(w_0)} \simeq \mathrm{Hom}_{Z(H)}\left(Z(N), \mathcal{S}_{w_0}\right) .$$

Proof: Let us put $U = \nu(K)$ so that $\nu^{-1}(U) = K \cdot N$ may be identified with $K \times_H N$, where $\left((wh^{-1}, h\beta)\right) = \left((w, \beta)\right)$ for all $w \in K$, $h \in H$ and $\beta \in N$. We use $\left((\alpha, \beta)\right)$ to denote a point in $\nu^{-1}(V) = K \times_H N$.

We must show that there is an isomorphism

$$\mathrm{map}\left(\nu^{-1}(U), S\right) \simeq \mathrm{Hom}_{Z(H)}\left(Z(N), \mathrm{map}(K, S)\right) .$$

We regard $\xi \in \mathrm{Hom}_{Z(H)}\left(Z(N), \mathrm{map}(K, S)\right)$ as an indexed family of maps $\xi_\beta : K \to S$ which satisfy $\xi_{h\beta}(w) = h_* \xi_\beta(wh)$ for all $h \in H$. Suppose first that we are given

$$f: K \times_H N \to S ,$$

then let

$$\xi_\beta(w) = \beta_* \left(f\left((w, \beta)\right)\right) .$$

Thus,

$$\begin{aligned} \xi_{h\beta}(w) &= h_* \beta_* \left(f\left((w\ h, \beta)\right)\right) \\ &= h_* \beta_* \left(f\left((wh, \beta)\right)\right) \\ &= h_* \xi_\beta(wh) . \end{aligned}$$

Conversely, if the $\{\xi_\beta\}$ are all given we put $f\left((w, \beta)\right) = \beta_*^{-1} \xi_\beta(w)$ and then

$$\begin{aligned} f\left((wh^{-1}, k\beta)\right) &= \beta_*^{-1} h_*^{-1} \xi_{h\beta}(wh^{-1}) \\ &= \beta_*^{-1} k_*^{-1}\left(h_* \xi_\beta(wh^{-1}h)\right) \\ &= \beta_*^{-1} \xi_\beta(w) . \end{aligned}$$

This shows that f is well defined. The $Z(N)$-module structure on $\mathrm{map}(K \times_H N, S)$ is given by

$$\alpha_{\#}(f)\big((w,\beta)\big) = \alpha_*\Big(f\big((w,\beta\alpha)\big)\Big) .$$

Thus

$$\beta_*\Big[\alpha_{\#}(f)\big((w,\beta)\big)\Big] = \beta_*\alpha_*\Big(f\big((w,\beta\alpha)\big)\Big) = \zeta_{\beta\alpha}(w) .$$

So we have exhibited a natural $Z(N)$-module isomorphism

$$\mathrm{Hom}_{Z(H)}\big(Z(N), \mathrm{map}(K,S)\big) \simeq \mathrm{map}\big(\nu^{-1}(U), S\big) .$$

The isomorphisms are natural and K may be taken arbitrarily small so the lemma follows.

Returning to the notation N_w for the isotropy subgroup at $w \in W$ we now have

(2.2) Theorem: At each point $w \in W$ there is a canonical isomorphism

$$h^j_{\nu(w)} \simeq H^j(N_w; \mathcal{S}_w) .$$

Proof: In view of (2.1) we need only apply Cartan-Eilenberg [10, prop. (7.4), p. 196] to conclude that $H^j(N_w, \mathcal{S}_w) \simeq H^j\Big(N; \mathrm{Hom}_{Z(N_w)}\big(Z(N), \mathcal{S}_w\big)\Big)$ for all $j \geq 0$.

For the group C^k the sheaf $\mathcal{S} \to W$ is $\mathcal{O}(W)^k \to V$, the sheaf of germs of nolomorphic maps on W into C^k.

(2.3) Corollary: The edge homomorphism $H^*\big(N; \mathcal{O}(W)^k\big) \to H^*(V; h^0_c)$ in the 'E-spectral sequence is an isomorphism.

Proof: In this case each stalk of $\mathcal{O}(W)^k$ is a complex vector space on which the finite group N_w acts as a group of complex linear automorphisms. Thus from (2.2) we see that $h^j_c \to V$ is the trivial sheaf if $j > 0$.

As a sheaf $h^0_c \to V$ is a module over $\mathcal{O}(V)$, the sheaf of germs of holomorphic functions $g: \nu^{-1}(U) \to C$ which satisfy $g(w\alpha) \equiv g(w)$, and on the other hand $h^0_c \to V$ arises by assigning to each open $U \subset V$ those holomorphic functions $f: \nu^{-1}(V) \to C^k$ which satisfy $f(w) \equiv \alpha_*\big(f(w\alpha)\big)$. Since α_* is complex linear the $\mathcal{O}(V)$-module structure on $h^0_c \to V$ is defined.

(2.4) Lemma: If for every isotropy subgroup $N_w \subset N$ the restriction of Φ to N_w is trivial then $h^0_c \to V$ is a locally free of rank k as a module over $\mathcal{O}(V)$.

Proof: We select $K \subset W$ as before so that $\nu^{-1}(U) = K \times_H N$. We denote by $E \subset \text{map}(K, C)$ the subalgebra of all holomorphic maps $g: K \to C$ with $g(wh) \equiv g(w)$. We wish to identify E^k with the submodule over E of all elements in $\text{map}(\nu^{-1}(U), C^k)$ which satisfy $\alpha_{\#}(f) = f$ for all $\alpha \in N$. Remember that h_* is the identity for all $h \in H$. If $f: K \times_H N \to C^k$ satisfies $\alpha_{\#}(f) = f$ then in particular $\alpha_* f((w, \alpha)) = f((w, e))$ and if $h \in H$, $h_* f((w, h)) = f((wh, e)) = f((w, e))$. Thus to each such f we may canonically associate $f((w, e)) \in E^k$. Conversely every element in E^k uniquely determines a map $f: \nu^{-1}(V) \to C^k$ satisfying $\alpha_{\#}(f) = f$. Thus locally $h_c^0 \to V$ is free of rank k over $\mathcal{O}(V)$.

If Z^{2k} is the group S, then $\mathcal{S} \to W$ is the constant sheaf $W \times Z^{2k} \to W$ acted on from the left by $(w, n) \to (w\alpha^{-1}, \alpha_*(n))$. We denote this sheaf with operators by $\mathfrak{Z}^{2k} \to W$. Let us denote by $\mathcal{T} \to W$ the sheaf of germs of holomorphic maps on W into T. From the short exact sequence

$$0 \to Z^{2k} \to C^k \to T \to 0$$

of $Z(N)$-modules we may deduce an exact cohomology triangle

$$\begin{tikzcd}[column sep=small] H^*(N; \mathfrak{Z}^{2k}) \arrow[rr] & & H^*(N; \mathcal{O}(W)^k) \arrow[dl] \\ & H^*(N; \mathcal{T}) \arrow[ul, "\delta"] & \end{tikzcd}$$

First we observe that K may be selected so that

$$0 \to \text{map}(K, Z^{2k}) \to \text{map}(K, C^k) \to \text{map}(K, T) \to 0$$

is a short exact sequence of $Z(H)$-modules. Since $Z(N)$ is a free left $Z(H)$-module it will follow from the proof of (2.1) that with $U = \nu(K)$, the sequence

$$0 \to \text{map}(\nu^{-1}(U), Z^{2k}) \to \text{map}(\nu^{-1}(U), C^k) \to \text{map}(\nu^{-1}(U), T) \to 0$$

is also a short exact sequence of left $Z(N)$-modules. Then clearly for each integer $j \geq 0$,

$$0 \to \text{Hom}_{Z(N)}(\beta_j(N), \text{map}(\nu^{-1}(U), Z^{2k})) \to \text{Hom}_{Z(N)}(\beta_j(N), \text{map}(\nu^{-1}(U), C^k))$$

$$\to \text{Hom}_{Z(N)}(\beta_j(N), \text{map}(\nu^{-1}(U), T)) \to 0$$

is exact. Exactness is preserved by direct limits, thus over V we obtain a short exact sequence of the differential sheaves

$$0 \to \mathcal{L}_Z \to \mathcal{L}_C \to \mathcal{L}_T \to 0 .$$

The cochains functor $C^i(V_j)$ preserves exactness so that we finally arrive at a short exact sequence of double complexes

$$0 \to L_Z^{\cdot\cdot} \to L_C^{\cdot\cdot} \to L_T^{\cdot\cdot} \to 0\ ,$$

to which we may apply the Kelly-Pitcher theorem to obtain the exact cohomology triangle. In view of (2.3) we shall write this as

$$\begin{tikzcd} H^*(N; \mathfrak{Z}^{2k}) \arrow[rr, "i_*"] & & H^*(V; h_c^0) \arrow[dl, "e_*"] \\ & H^*(N; \mathfrak{T}) \arrow[ul, "\delta"] & \end{tikzcd}\ .$$

3. The Groups $H^*(N; \mathfrak{Z}^{2k})$

We shall restrict our attention to the sheaf with operators $\mathfrak{Z}^{2k} \to W$ in this section. From (2.2) we immediately see in this case that for every $w \in W$

$$H_\phi^*(N_w; Z^{2k}) \simeq h_{\nu(w)}^*$$

where the cohomology of N_w is formed from the $Z(N_w)$-module structure on Z^{2k} which is defined by the restricted homomorphism $\phi_w : N_w \to \mathrm{Aut}(T)$. Regarding $0 \to Z^{2k} \to C^k \to T \to 0$ as an exact sequence of $Z(N_w)$-modules and using the fact that N_w is finite we find that $\delta : H_\phi^j(N_w; T) \simeq H_\phi^{j+1}(N_w; Z^{2k})$ for all $j > 0$. In particular, $H_\phi^1(N_w; T) \simeq H_\phi^2(N_w; Z^{2k})$. An element of $H_\phi^2(N_w; Z^{2k})$ is the Baer class of a group extension

$$0 \to Z^{2k} \to \pi_w \to N_w \to 1$$

which is compatible with $\phi_w : N_w \to \mathrm{Aut}(T)$. On the other hand, a 1-cocycle $\chi \in Z_\phi^1(N_w; T)$ is a crossed homomorphism; that is a function $\chi : N_w \to T$ which satisfies $\chi(\alpha\beta) = \chi(\alpha)\alpha_*\big(\chi(\beta)\big)$. We may use χ to define a right action (T, N_w) by $t * \alpha = \alpha_*^{-1}(t)\chi(\alpha^{-1})$. To see that this is a right action we write $(t * \alpha) * \beta = \beta_*^{-1}\alpha_*^{-1}(t)\beta_*^{-1}\big(\chi(\alpha^{-1})\big)\,\chi(\beta^{-1})$. Since T is abelian

$$\chi\big((\alpha\beta)^{-1}\big) = \chi(\beta^{-1}\alpha^{-1}) = \chi(\beta^{-1})\beta_*^{-1}\big(\chi(\alpha^{-1})\big) = \beta_*^{-1}\big(\chi(\alpha^{-1})\big)\,\chi(\beta^{-1})\ .$$

Thus $(t * \alpha) * \beta = t * (\alpha\beta)$. The next lemma is a known result about Bieberbach groups [2, Th. 1].

(3.1) Lemma: The right action (T, N_w) is principal if and only if the group extension π_w corresponding to $\delta(c\ell(\chi)) \in H^2_\Phi(N_w Z^{2k})$ is torsionless.

Proof: Suppose first that there is a non-trivial subgroup $H \subset N_w$ which leaves a point $t_0 \in T$ fixed. For all $\alpha \in H$ we have $\alpha_*(t_0)\chi(\alpha) = t_0$, or $\chi(\alpha) = t_0\alpha_*(t_0^{-1})$ for all $\alpha \in H$. This is a principal crossed homomorphism and thus $c\ell(\chi)$ lies in the kernel of $H^1_\Phi(N_w; T) \to H^1_\Phi(H; T)$. Since δ is natural it follows that $0 \to Z^{2k} \to \pi_w \to N_w \to 1$ splits over the subgroup H; that is, π_w contains a subgroup isomorphic to the semi-direct product $Z^{2k} \circ H$. It follows that π_w contains torsion if (T, N_w) is not a principal action. Conversely, if (T, N_w) is principal then π_w is the fundamental group of the quotient T/N_w, so of course π_w is torsionless. When π_w is torsionless it is by definition a Bieberbach group [11], but the image of $Z^{2k} \to \pi_w$ may not be the maximal normal abelian subgroup so that in general N_w need not be the holonomy group.

Let us next consider the edge homomorphism of the 'E-spectral sequence, $H^2(N; \mathfrak{z}^{2k}) \to H^0(V; h^2)$ together with the isomorphisms, $H^2_\Phi(N_w; Z^{2k}) \simeq h^2_{i(w)}$. To any element in $H^2(N; \mathfrak{z}^{2k})$ then there is associated a whole family of group extensions $0 \to Z^{2k} \to \pi_w \to N_w \to 1$, one for each point of W. If w is replaced by $w\alpha$ then N_w is replaced by $\alpha^{-1}N_w\alpha = N_{w\alpha}$ and $\Phi_{w\alpha}(\alpha^{-1}h\alpha) = \Phi(\alpha^{-1})\Phi_w(h)\Phi(\alpha)$ for all $h \in N_w$. Thus $\pi_{w\alpha}$ is canonically isomorphic to π_w.

(3.2) Definition: An element of $H^2(N; \mathfrak{z}^{2k})$ is a Bieberbach class if and only if at every point $w \in W$ the induced group extension π_w is torsionless. The subset of Bierberbach classes is denoted by $\mathcal{B}_\Phi \subset H^2(N; \mathfrak{z}^{2k})$.

The set $\mathcal{B}_\Phi$ may of course be empty, however if it contains at least one Bieberbach class then it contains more elements than might at first be apparent. Consider the other edge homomorphism of the 'E-spectral sequence,

$$H^2(V; h^0) \to H^2(N; \mathfrak{z}^{2k}) .$$

By definition, the composition of the two edge homomorphisms

$$H^2(V; h^0) \to H^2(N; \mathfrak{z}^{2k}) \to H^0(V; h^2)$$

is trivial, thus we have

(3.3) Lemma: The sum of any Bieberbach class with an element in the image of the edge homomorphism

$$H^2(V; h^0) \to H^2(N; \mathfrak{z}^{2k})$$

is still a Bieberbach class.

Thus we can translate a Bieberbach class by an element from $H^2(V; h^0)$. By analogy with (2.4) we can state

(3.4) Lemma : If for each isotropy subgroup the restriction $\phi|N_w \to \mathrm{Aut}(T)$ is trivial, then $h^0 \to V$ is a locally constant coefficient system, locally isomorphic to Z^{2k}. If ϕ itself is trivial then $H^*(V; h^0) \simeq H^*(V; Z^{2k})$.

Related to (3.3) and (3.4) we have

(3.5) Lemma: If at each $w \in W$ the restriction $\phi|N_w \to \mathrm{Aut}(T)$ is trivial then

$$0 \to H^2(V; h^0) \to H^2(N; \mathfrak{z}^{2k}) \to H^0(V; h^2) \xrightarrow{d_3} H^3(V; h^0) \to H^3(N; \mathfrak{z}^{2k})$$

is exact. Furthermore, if $\mathcal{B}_\phi$ is non-empty then every isotropy subgroup N_w is isomorphic to a subgroup of T.

Proof: Since $\phi_w : N_w \to \mathrm{Aut}(T)$ is trivial we have from (2.2)

$$h^1_{\nu(w)} \simeq H^1(N_w; Z^{2k}) = \mathrm{Hom}(N_w; Z^{2k}) = 0$$

because N_w is finite and the exactness follows by the usual spectral sequence considerations.

If $\mathcal{B}_\phi$ is not empty then for each N_w there is a homomorphism $\chi : N_w \to T$ for which $t * \alpha = t\chi(\alpha^{-1})$ defines a principal action (T, N_w). Clearly such a χ must be a monomorphism.

We should also indicate

(3.6) Lemma: If W is simply connected then there is a $1-1$ correspondence between the elements of $\mathrm{Hom}(\pi_1(V), \mathrm{Aut}(T))$ and the subset of $\mathrm{Hom}(N, \mathrm{Aut}(T))$ consisting of those homomorphisms which annihilate all of the isotropy subgroups.

Proof: The quotient of N by the least normal subgroup containing all the isotropy groups is isomorphic to $\pi_1(V)$, [1].

Now we turn briefly to the "E-spectral sequence. Since $\mathfrak{z}^{2k} \to W$ is just $W \times Z^{2k} \to$ acted on by $\alpha(w, n) = (w\alpha^{-1}, \alpha_*(n))$, $H^*(W; \mathfrak{z}^{2k})$ is $H^*(W; Z^{2k})$ with the induced

$Z(N)$-module structure. Thus $''E_2^{i,j} \simeq H^i\big(N; H^j(W; Z^{2k})\big)$, and in particular $''E_2^{i,j} \simeq H^i_\Phi(N; Z^{2k})$.

(3.7) Definition: Let $A_\Phi \subset H^2_\Phi(N; Z^{2k})$ be the set of group extensions

$$0 \to Z^{2k} \to \pi \to N \to 1$$

for which every induced extension $0 \to Z^{2k} \to \pi_w \to N_w \to 1$ is torsionless.

The set A_Φ is carried into $\mathcal{B}_\Phi$ by the edge homomorphism

$$H^2_\Phi(N; Z^{2k}) \to H^2(N; \mathfrak{z}^{2k}) .$$

By analogy with (3.5) we state

(3.8) Lemma: If W is simply connected then $H^1_\Phi(N; Z^{2k}) \simeq H^1(N; \mathfrak{z}^{2k})$ and

$$0 \to H^2_\Phi(N; Z^{2k}) \to H^2(N; \mathfrak{z}^{2k}) \to H^0\big(N; H^2(W; Z^{2k})\big) \xrightarrow{d_3} H^3_\Phi(N; Z^{2k}) \to H^3(N; \mathfrak{z}^{2k}).$$

is exact.

(3.9) Lemma: If W is 2-connected then $H^2_\Phi(N; Z^{2k}) \cong H^2(N; \mathfrak{z}^{2k})$ and $A_\Phi \cong B_\Phi$.

If W is contractible then $H^*_\Phi(N; Z^{2k}) \simeq H^*(N; \mathfrak{z}^{2k})$ and the set $A_\Phi \simeq \mathcal{B}_\Phi$ consists of all group extensions $0 \to Z^{2k} \to \pi \to N \to 1$ for which π is torsionless.

Proof: The cohomological part of the assertion follows immediately from the $''E$-spectral sequence. Suppose now that W is contractible and in π there is an element of prime order. We must show that this element of prime order is contained in π_w for some $w \in W$. The image of this element in N also has prime order. Since W is contractible it follows from Smith theory that this image element has a fixed point and hence lies in some N_w. Of course the original element is contained in π_w. We have thus shown that $0 \to Z^{2k} \to \pi \to N \to 1$ is an element of A_Φ if and only if π is torsionless.

We shall discuss an alternate description of $H^*(N; \mathfrak{z}^{2k})$ in terms of the Borel space. Thus we take (N, X) to be a left principal action of N on a contractible complex so that $X/N = K(N,1)$. The Borel space then is $Y = W \times_N X$ where $\big((w,x)\big) = \big(w\alpha^{-1}, \alpha x\big)$ for all

$\alpha \in N$. We use $((w,x))$ to denote a point in Y. Over Y we introduce a local coefficient bundle $\mathfrak{Z}^{2k} \to Y$ by letting N act from the left on $W \times X \times Z^{2k}$ where $\alpha(w,x,n) = (w\alpha^{-1}, \alpha x, \alpha_*(n))$ and setting $\mathfrak{Z}^{2k} = (W \times X \times Z^{2k})/N$. It is intuitively clear that $H^*(N; \mathfrak{Z}^{2k}) \simeq H^*(Y; \mathfrak{Z}^{2k})$, thus we omit the proof which is quite tedious. There are maps

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ W/N = V & & X/N = K(N,1) \end{tikzcd}$$

the second of which is a fibration with fibre W and structure group N. If, with respect to $\mathfrak{Z}^{2k} \to Y$, the Leray spectral sequences of these two maps are introduced, then we obtain $'E$ and $''E$ as in § 2.

We shall use this representation of $H^*(N; \mathfrak{Z}^{2k})$ to study the following situation. Suppose that $L \subset N$ is a normal subgroup for which

(i) L *acts freely on* W

(ii) L *lies in the kernel of* ϕ.

We denote by $q: N \to N/L = G$ the quotient homomorphism and by $\mu: W \to W/L = B$ the quotient map with respect to the action of L. Since (W, L) is a properly discontinuous group of covering transformations, the quotient space B is also a complex analytic manifold. There is induced a properly discontinuous holomorphic action (B, G) such that $\mu(w\alpha) = \mu(w)q(\alpha)$. Since $L \cap N_w = \{e\}$ the restriction $q/N_w \to G$ is an isomorphism onto $G_{\mu(w)}$. Since L lies in the kernel of ϕ there is induced a homomorphism $\psi: G \to \mathrm{Aut}(T)$. Now with respect to $\mathfrak{Z}^{2k} \to B$ we may also consider $H^*(G; \mathfrak{Z}^{2k})$.

(3.10) *Theorem*: *If* $L \subset N$ *is a normal subgroup satisfying* (i) *and* (ii) *then*

$$H^*(G; \mathfrak{Z}^{2k}) \simeq H^*(N; \mathfrak{Z}^{2k}) .$$

Proof: Let (G, X') be a left principal action on a contractible complex so that $X'/G = K(G,1)$. Consider then the Borel space of (W, L); that is, $W \times_L X$. A right action $(W \times_L X, G)$ is defined as follows. If $\alpha \in N$ then

$$[w,x]q(\alpha) = [w\alpha, \alpha^{-1}x] .$$

If $\beta \in L$ then $[w\beta^{-1}, \beta x] = [w,x]$, but $[w\beta^{-1}\alpha, \alpha^{-1}\beta x] = [w\alpha(\alpha^{-1}\beta^{-1}\alpha), (\alpha^{-1}\beta\alpha)\alpha^{-1}w]$ $= [w\alpha, \alpha^{-1}w]$. The subgroup $L \subset N$ acts trivially so that $(W \times_L X, G)$ is well defined. Next we form the Borel space of $(W \times_L X, G)$; that is, $(W \times_L X) \times_G X'$. A point in this space is

denoted by $(([w,x], x'))$. There are maps

$$\begin{tikzcd} & (W \times_L X) \times_G X' \arrow[dl] \arrow[dr] & \\ W \times_N X & & B \times_G X' \end{tikzcd}$$

given by

$$(([w,x], x')) \to ((w,x))$$

$$(([w,x], x')) \to (\mu(w), x') \quad .$$

The first map is a fibration with structure group G and fibre the contractible complex X' Because L acted freely on W, the second map is also a fibration with structure group N and fibre the contractible complex X. Thus the Borel spaces of (B, G) and of (W, N) are of the same homotopy type. Since L lies in the kernel of ϕ we may define a local coefficient bundle $\hat{\mathfrak{z}}^{2k} \to (W \times_L X) \times_G X'$ by allowing G to act on $(W \times_L X) \times_G X' \times Z^{2k}$ with

$$g(\alpha)\left(([w,x], x'), n\right) = \left(([w\alpha^{-1}, \alpha x], g(\alpha)x'), \alpha_*(n)\right)$$

and setting $\hat{\mathfrak{z}}^{2k} = ((W \times_L X) \times X'/G)$. There are the bundle maps

$$\begin{tikzcd} \mathfrak{z}^{2k} \arrow[d] & \hat{\mathfrak{z}}^{2k} \arrow[l] \arrow[r] \arrow[d] & \mathfrak{z}^{2k} \arrow[d] \\ W \times_N X & (W \times_L X) \times_G X' \arrow[l] \arrow[r] & B \times_G X' \end{tikzcd}$$

wherein the maps on the base spaces are homotopy equivalences. Thus

$$H^*(N; \mathfrak{z}^{2k}) \simeq H^*\left((W \times_L X) \times_G X', \hat{\mathfrak{z}}^{2k}\right)$$

$$\simeq H^*(G; \mathfrak{z}^{2k}) \; .$$

This concludes the proof of (3.10).

We shall apply this to the universal covering action (W^*, N^*) associated to (W, N). There is a canonical group extension

$$1 \to \pi_1(W) \to N^* \to N \to 1$$

and a properly discontinuous holomorphic action on the universal covering space W^* so that the covering map $W^* \to W$ is equivariant with respect to the quotient homomorphisms $N^* \to N$. The construction of N^* and its action on W^* is fully discussed in [17] so we shall only give a brief summary here. Choose a base point $w \in W$ and let $P(w, W)$ be the space of paths in W issuing from w. For each $\alpha \in N$ choose a path $P_\alpha(\tau) \in P(w, W)$ with $P_\alpha(1) = w\alpha$, agreeing that $P_e(\tau)$ is the trivial path at w. To each $\alpha \in N$ we associate an automorphism of $\pi_1(W, w)$, denoted as usual by α_*. If $\sigma \in \pi_1(W, w)$ is represented by the closed loop $\ell(\tau)$ at w then $\alpha_*(\sigma)$ is represented by

$$P_\alpha(3\tau), \quad 0 \le \tau \le 1/3$$

$$\ell(3\tau - 1)\alpha, \quad 1/3 \le \tau \le 2/3$$

$$P_\alpha(3 - 3\tau), \quad 2/3 \le \tau \le 1 .$$

Next we define the non-abelian extension cocycle $f: N \times N \to \pi_1(W, w)$. For any pair (α, β) we take $f(\alpha, \beta)$ to be the element of the fundamental group represented by

$$P_\beta(3\tau), \quad 0 \le \tau \le 1/3$$

$$P_\alpha(3\tau - 1)\beta, \quad 1/3 \le \tau \le 2/3$$

$$P_{\alpha\beta}(3 - 3\tau), \quad 2/3 \le \tau \le 1 .$$

When the product in $\pi_1(W, w)$ is defined as in [13 ; § 2], that is, for two closed loops $\ell_1(\tau), \ell_2(\tau)$ at w

$$(\ell_1 \circ \ell_2)(\tau) = \begin{cases} \ell_2(\tau), & 0 \le \tau \le 1/2 \\ \ell_1(2\tau - 1), & 1/2 \le \tau \le 1 \end{cases}$$

then a group structure on $N^* = N \times \pi_1(W, w)$ is given by

$$(\alpha, \sigma_1)(\beta, \sigma_2) = \left(\alpha\beta,\ f(\alpha, \beta)\, \beta_*(\sigma_1)\sigma_2\right) .$$

Now N^* must act on the universal covering manifold W^*. Suppose $b \in W^*$, $(\alpha, \sigma) \in N^*$, then let $p(\tau) \in P(w, W)$ be a path representing b. The effect of acting on b with (α, σ) is the point of W^* represented by the path in $P(w, W)$:

$$\ell(3\tau),\ 0 \le \tau \le 1/3$$

$$P_\alpha(3\tau-1),\ 1/3 \le \tau \le 2/3$$

$$p(3\tau-2)\alpha,\ 2/3 \le \tau \le 1 .$$

This yields a well defined properly discontinuous group of holomorphic transformations (W^*, N^*) covering (W, N). A special case worth noting is a finite group N acting on W with a fixed point $w \in W$ chosen as the base point. We then may take all paths $P_\alpha(\sim)$ to be trivial and N^* is a semi-direct product $N \circ \pi_1(W, w)$ with product $(\alpha, \sigma_1)(\beta, \sigma_2) = \left(\alpha\beta,\ \beta_*(\sigma_1)\sigma_2\right)$.

In any case $\Phi : N \to \mathrm{Aut}(T)$ induces a homomorphism also denoted by $\Phi : N^* \to \mathrm{Aut}(T)$. As an immediate corollary of (3. 10) we obtain

(3. 11) Corollary : There is an isomorphism $H^*(N; \mathfrak{Z}^{2k}) \simeq H^*(N^*; \mathfrak{Z}^{2k})$.

This implies that there is no real loss of generality in assuming that W is simply connected.

(3. 12) Corollary: If W is aspherical then

$$H^*(N; \mathfrak{Z}^{2k}) \simeq H^*_\Phi(N^*; Z^{2k}) .$$

Since W is aspherical W^* must be contractible so we apply (3. 11) and (3. 9).

We shall close the section now with one remark.

(3. 13) Lemma : If (W, N) is a finite group acting with at least one fixed point then $\mathcal{B}_\Phi \subset H^2(N; \mathfrak{Z}^{2k})$ is non-empty if and only if there is a torsion-less extension $0 \to Z^{2k} \to \pi \to N \to 1$ in $H^2_\Phi(N; Z^{2k})$.

The proof is left to the reader, but we remember from (3. 3) that we may be able to expand on an element in $\mathcal{B}_\Phi$.

4. The group $H^1(N;\mathfrak{J})$

Let us now discuss the geometric significance of

$$\to H^1(V,h_c^0) \xrightarrow{e_*} H^1(N;\mathfrak{J}) \xrightarrow{\delta} H^2(N;\mathfrak{Z}^{2k}) \xrightarrow{i_*} H^2(V;h_c^0) \longrightarrow \quad .$$

For each element in $H^1(N;\mathfrak{J})$ we shall construct a $(T,B,N) \to (W,N)$; that is, a left principal holomorphic T-bundle with right operators N. The actions of T and N will be related by the formula

$$(tb)\alpha = \alpha_*^{-1}(t)(b\alpha) \ .$$

We shall see that N acts freely on B as a properly discontinuous group of holomorphic covering transformations if and only if $\delta(\tau) \in H^2(N;\mathfrak{Z}^{2k})$ is a Bieberbach class.

In such cases we shall of course form the quotient $M = B/N$ and obtain a diagram

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ M_\tau \arrow[dr, "\psi"'] & & W \arrow[dl, "\nu"] \\ & V & \end{tikzcd}$$

so that M_τ is a non-singular complex analytic manifold and $\psi: M_\tau \to V$ is a holomorphic Seifert fibration. The generic fibre is T. If N_w is non-trivial, however, $\psi^{-1}(\nu(w))$ is the non-singular manifold T/N_w where the group extension $0 \to Z^{2k} \to \pi_1(T/N_w) \to N_w \to 0$ is given by the element of $H^2_\phi(N_w; Z^{2k}) \simeq h^2_\nu(w)$ which is determined by $\delta(\tau)$.

If $\phi: N \to \mathrm{Aut}(T)$ annihilates all the isotropy subgroups of (W,N) then $\psi: M_\tau \to V$ is a "holomorphic local action" of T on M_τ, [27]. Finally if ϕ is trivial then the actions of T and N on B will comute so that we receive a holomorphic action (T, M_τ) with $M_\tau/T = V$. If V is compact then M_τ is closed; if W is aspherical then so is M_τ.

Suppose $a \in \mathcal{B}_\phi$ is some Bieberbach class. It has a holomorphic realization (lies in the image of δ) if and only if $i_*(a) = 0 \in H^2(V;h_c^0)$. If ϕ annihilates all the isotropy subgroups then by (2.4) $h_c^0 \to V$ is locally free so it may be regarded as the sheaf of germs of holomorphic sections of a complex vector bundle over V. The bundle is found as follows. Let $\phi: N \to \mathrm{Aut}(T)$ induce $\phi: N' \to \mathrm{Aut}(T)$ and then appeal to (3.6) for the homomorphism $\pi_1(V) \to \mathrm{Aut}(T) \subset GL(k, C)$. The bundle over V is then constructed as usual. When V is a compact analytic space $H^*(V;h_c^0)$ will then be finite dimensi onal.

On the other hand, a Bieberbach class need not have a unique holomorphic representation for if $v \in H^1(V;h_c^0)$ then $\delta(\tau + e_*(v)) = \delta(\tau)$. Since $H^1(V;h_c^0)$ is a linear space we can

regard $\Phi: M_{\tau+e_*(v)} \rightarrow V$ as a deformation of the holomorphic Seifert fibration $\Phi: M_\tau \rightarrow V$. No topological change is involved.

Let $\mathcal{U} = \{U_i\}$ be a locally finite indexed open covering of V and put $W_i = \nu^{-1}(U_i)$ so that the $\{W_i\}$ form an open cover of W by N-invariant open sets. A holomorphic co-ordinate system with operators $\{m_{i,j}(w, \alpha)\}$ is the assignment to each ordered pair (i, j), with $U_i \cap U_j \neq \emptyset$, of a holomorphic map

$$m_{i,j}:(W_i \cap W_0) \times N \rightarrow T$$

which for every ordered triple (i, j, k), with $U_i \cap U_j \cap U_k \neq \emptyset$ satisfies on $(W_i \cap W_j \cap W_k) \times N$ the cocycle condition

$$m_{i,k}(w, \alpha\beta) = m_{i,j}(w, \alpha)\alpha_*\left(m_{j,k}(w\alpha, \beta)\right) \quad . \tag{4.1}$$

In particular

$$m_{i,k}(w, e) = m_{i,j}(w, e) m_{j,k}(w, e)$$

so that the $\{m_{i,j}(w, e)\}$ form as usual a holomorphic co-ordinate system for a principal T-bundle over W. An important consequence of the cocycle condition is the identity

$$m_{i,k}\left(w\alpha\beta, (\alpha\beta)^{-1}\right) = m_{i,j}(w\alpha\beta, \beta^{-1})\beta_*^{-1}\left(m_{j,k}(w\alpha, \alpha^{-1})\right) . \tag{4.2}$$

This equation will define the right action of N on B.

Let us first form in the standard fashion [34; sec. 3], a principal holomorphic co-ordinate T-bundle over W. We take the disjoint union $\bigcup (T \times W_i \times \{i\})$ and introduce the equivalence relation $(t, w, i) \sim (t', w', j)$ if and only if $w = w'$ and $t' m_{i,j}(w, e) = t$. The resulting bundle space is denoted by B and a point by $b = ((t, w, i))$. The projection $p: B \rightarrow W$ is $p((t, w, i)) = w$. The left action (T, B) is given by $t_1 ((t, w, i)) = ((t_1 t, w, i))$.

Now we shall define the right action (B, N). We put

$$((t, w, i))\alpha = \left(\left(\alpha_*^{-1}(t)\, m_{i,i}(w\alpha, \alpha^{-1}),\ w\alpha, i\right)\right) .$$

We must show that this is well defined and does yield a right action of N. Suppose then that $((t, w, i)) = ((t', w, j))$, so that $t' m_{i,j}(w, e) = t$. First we apply (4.2) to the triple (i, j, j) with $\beta = e$ to obtain the equation

$$m_{i,j}(w\alpha, \alpha^{-1}) = m_{i,j}(w\alpha, e) m_{j,j}(w\alpha, \alpha^{-1}).$$

Then, with the roles of α and β interchanged, we apply (4.2) to (i, i, j), yielding

$$m_{i,j}(w\alpha, \alpha^{-1}) = m_{i,i}(w\alpha, \alpha^{-1})\alpha_*^{-1}\left(m_{i,j}(w, e)\right).$$

Finally then we arrive at the identity

$$(4.3) \qquad m_{i,j}(w\alpha, e)m_{j,j}(w\alpha, \alpha^{-1}) = m_{i,i}(w\alpha, \alpha^{-1})\alpha_*^{-1}\left(m_{i,j}(w, e)\right) \quad .$$

Now we can write

$$\alpha_*^{-1}(t)m_{i,i}(w\alpha, \alpha^{-1}) = \alpha_*^{-1}(t')\alpha_*^{-1}\left(m_{i,j}(w, e)\right) m_{i,i}(w\alpha, \alpha^{-1})$$

$$= \left(\alpha_*^{-1}(t')m_{j,j}(w\alpha, \alpha^{-1})\right) m_{i,j}(w\alpha, e) \quad .$$

Thus we have shown that our action is well defined, so next we must use (4.2) to prove that the composition rule is satisfied. Let us write

$$\left(\left(\alpha_*^{-1}(t)m_{i,i}(w\alpha, \alpha^{-1}), w\alpha, i\right)\right)\beta$$

$$= \left(\left(\beta_*^{-1}\alpha_*^{-1}(t)\beta_*^{-1}\left(m_{i,i}(w\alpha, \alpha^{-1})\right) m_{i,i}(w\alpha\beta, \beta^{-1}), w\alpha\beta, i\right)\right)$$

$$= \left(\left((\alpha\beta)_*^{-1}(t)m_{i,i}\left(w\alpha\beta, (\alpha\beta)^{-1}\right), w\alpha\beta, i\right)\right)$$

$$= \left((t, w, i)\right)\alpha\beta \quad .$$

Therefore we have defined (B, N). Obviously $(B, N) \twoheadrightarrow (W, N)$ is equivariant and $(tb)\alpha = \alpha_*^{-1}(t)(b\alpha)$.

Let us turn now to the matter of equivalence of co-ordinate systems with operators. We shall say that $\left\{M_{i,j}(w, \alpha)\right\}$ is equivalent to $\left\{m_{i,j}(w, \alpha)\right\}$ if and only if there is for each index i a holomorphic map $\zeta_i : W_i \to T$ which for each (i, j), with $U_i \cap U_j \neq \emptyset$, satisfies

$$M_{i,j}(w, \alpha) = m_{i,j}(w, \alpha)\alpha_*\left(\zeta_j(w\alpha)\right)\zeta_i(w)^{-1}.$$

Suppose that (T, B', N) was formed from $\left\{M_{i,j}(w, \alpha)\right\}$. If we note that

$$M_{i,i}(w\alpha, \alpha^{-1})\zeta_i(w\alpha) = m_{i,i}(w\alpha, \alpha^{-1})\alpha_*^{-1}\left(\zeta_i(w)\right)$$

then we may define a T-N equivariant holomorphic equivalence

$$\begin{tikzcd}
(T, B', N) \arrow[r] \arrow[dr] & (T, B, N) \arrow[d] \\
& (W, N)
\end{tikzcd}$$

by

$$\big((t, w, i)\big)' \to \big((t\,\zeta_i(w),\, w,\, i)\big) \;.$$

Since $M_{i,j}(w, e)\zeta_i(w) = m_{i,j}(w, e)\zeta_j(w)$ this is well defined an obviously T-equivariant. It is also N-equivariant because

$$\Big(\big(\alpha_*^{-1}(t)M_{i,i}(w\alpha, \alpha^{-1}), w\alpha, i\big)\Big)' \longrightarrow \Big(\big(\alpha_*^{-1}(t)M_{i,i}(w\alpha, \alpha^{-1})\zeta_i(w\alpha), w\alpha, i\big)\Big)$$

$$= \Big(\Big(\alpha_*^{-1}\big(t\zeta_i(w)\big)\, m_{i,i}(w\alpha, \alpha^{-1}), w\alpha, i\Big)\Big)$$

$$= \Big(\big(t\zeta_i(w), w, i\big)\Big)\alpha \quad .$$

We may conclude, thus, that (T, B, N) <u>only depends on the equivalence class of</u> $\big\{m_{i,j}(w, \alpha)\big\}$.

Let us relate the introductory considerations in section 3 to the co-ordinate systems with operators. To each equivalence class we can canonically associate a characteristic family of elements $\chi_w \in H^1_\Phi(N_w; T)$. If $w \in W_i$ we define

$$\chi_{w,i} : N_w \to T$$

by $\chi_{w,i}(\alpha) = m_{i,i}(w, \alpha)$. If $w\beta = w$ also then $\chi_{w,i}(\alpha, \beta) = m_{i,i}(w, \alpha\beta) =$ $m_{i,i}(w, \alpha)\alpha_*\big(m_{i,i}(w, \beta)\big) = \chi_{w,i}(\alpha)\alpha_*\big(\chi_{w,i}(\beta)\big)$. Thus each $\chi_{w,i}$ is a crossed homomorphism. If $w \in W_j$ also, then since $w\alpha = w$ we have from (4.1)

$$m_{i,j}(w, \alpha) = m_{i,j}(w, e)\chi_{w,j}(\alpha)$$

$$m_{i,j}(w, \alpha) = \chi_{w,i}(\alpha)\alpha_*\big(m_{i,j}(w, e)\big)$$

so that putting $t_0^{-1} = m_{i,j}(w, e)$ we have

$$\chi_{w,j}(\alpha) = \chi_{w,i}(\alpha)t_0\alpha_*(t_0^{-1}) \;.$$

Thus $c\ell(\chi_w) \in H^1_\Phi(N_w; T)$ is independent of the neighborhood containing w. If $m_{i,i}(w, \alpha)$ is replaced by $m_{i,i}(w, \alpha)\alpha_*\big(\zeta_i(w\alpha)\big)\zeta_i(w)^{-1}$ then $\chi_{w,i}(\alpha)$ is replaced by $\chi_{w,i}(\alpha)\alpha_*\big(\zeta_i(w)\big)\zeta_i(w)^{-1}$, but with w fixed $\alpha \to \alpha_*\big(\zeta_i(w)\big)\zeta_i(w)^{-1}$ is a principal crossed homomorphism of N_w into T.

Thus we finally conclude that $c\ell(\chi_w) \in H^1_\Phi(N_w; T)$ depends on the equivalence class of the holomorphic co-ordinate system with operators.

We are really concerned with the cases in which (B,N) acts freely. Clearly the necessary and sufficient condition is that for each isotropy subgroup the action (T, N_w) given by $t * \alpha = \alpha_*^{-1}(t)\chi_{w,i}(\alpha^{-1})$ be a principal action. Hence, following (3.1), we may state

(4.4) <u>Lemma</u>: <u>The action</u> (B,N) <u>is free if and only if at each</u> $w \in W$ <u>the group extension</u> $0 \to Z^{2k} \to \pi_w \to N_w \to 1$ <u>associated with</u> $\delta(\chi_w) \in H^2_\Phi(N_w; Z^{2k})$ <u>is torsionless</u>.

In this case we Put $M = B/N$ so that M is a non-singular complex analytic manifold. A point in M is denoted by $[t,w,i]$. The holomorphic Seifert fibration $\Psi: M \to V$ is given by $\Psi([t,w,i]) = v(w) \in V$ and $\Psi^{-1}(\nu(w)) = T/N_w$.

Actually we are discussing a <u>Čech Theory definition</u> of $H^1(N; \mathcal{J})$. Let us take up this point in detail. Associated to each locally finite open covering $\mathcal{U}$ of F there is the nerve $N(\mathcal{U})$ which may be regarded as a category. An object (simplex) is a finite set of elements in $\mathcal{U}$ with non-empty intersection. The morphisms are the inclusions. If σ is a simplex then $1+\dim(\sigma)$ is the cardinality of the set σ. If $\sigma = (U^0, \ldots, U^p)$ then $\mathrm{Sup}(\sigma)$ is $\iota^{-1}(U^0) \cap \ldots \cap \nu^{-1}(U^p) \subset W$. Thus $\sigma \to \mathrm{Sup}(\sigma)$ is a contravariant functor from $N(\mathcal{U})$ to the category of open non-empty subsets of W, with morpnisms the inclusions. Using $\mathrm{Sup}(\cdot)$ we have a covariant functor $\sigma \to \mathrm{map}(\mathrm{Sup}(\sigma), T)$ into the category of left $Z(N)$-modules. We wish to define the cohomology of N with coefficients in this covariant stack with operators.

To each simplex $\sigma \in N(\mathcal{U})$ there is associated the simplicial chain group $\{C_p(\sigma), d_j, s_i\} = C(\sigma)$. Thus $C_p(\sigma)$ is the free abelian group generated by all the ordered $(p+1)$-tuples of vertices of σ. The face and degeneracy operators are defined as usual. At the same time there are defined face and degeneracy operators in the bar resolution so that $\beta(N) = \{\beta_p(N), d'_j, s'_i\}$ is a simplicial $Z(N)$-module, [25, p. 233-238]. Define a covariant functor on $N(\mathcal{U})$ to the category of simplicial $Z(N)$-modules by forming the <u>cartesian product</u> $\beta(N) \times C(\sigma) = \{\beta_p(N) \otimes C_p(\sigma),\ d'_j \otimes d_j,\ s'_i \otimes s_i\}$.

A p-cochain is a natural transformation from the functor $\sigma \to \beta_p(N) \otimes C_p(\sigma)$ to the functor $\sigma \to \mathrm{map}(\mathrm{Sup}(\sigma), T)$. The p-cochains form an abelian group $C^p_{\mathcal{U}}(N; \mathcal{J})$. Since the boundary operator is a natural transformation of $\beta_{p+1}(N) \otimes C_{p+1}(\sigma)$ to $\beta_1(N) \otimes C_p(\sigma)$ we immediately obtain a coboundary operator $\delta: C^p_{\mathcal{U}}(N; \mathcal{J}) \to C^{p+1}_{\mathcal{U}}(N; \mathcal{J})$. In this way $H^*_{\mathcal{U}}(N; \mathcal{J})$ is defined.

We assert that $H^1_{\mathcal{U}}(N;\mathcal{J})$ is the group of equivalence classes of co-ordinate systems with operators. According to naturality a 1-cochain in $C^1_{\mathcal{U}}(N;\mathcal{J})$ is uniquely determined by a function which to each ordered triple $(\alpha; U_i, U_j)$, $\alpha \in N$, $U_i \cap U_j \neq \emptyset$, assigns an element $c(\alpha; U_i, U_j) \in \text{map}(W_i \cap W_j, T)$. Following this view, δc is evaluated on each $(\alpha, \beta; U_i, U_j, U_k)$, where $U_i \cap U_j \cap U_k \neq \emptyset$, and $\delta c(\alpha, \beta; U_i, U_j, U_k) = \alpha_{\#}\big(c(\beta; U_j, U_k)\big)\big(c(\alpha\beta; U_i, U_k)\big)^{-1}\big(c(\alpha; U_i, U_j)\big)$ in $\text{map}(W_i \cap W_j \cap W_k, T)$. Thus if $m_{i,j}(w, \alpha)$ is the value of the map $c(\alpha; U_i, U_j)$ at a point $w \in W_i \cap W_j$ then we see that $\{m_{i,j}(w, \alpha)\}$ is a co-ordinate system with operators if and only if $c(\alpha; U_i, U_j)$ is a 1-cocycle. If $[\ ] \in \beta_0(N)$ is the generator of $\beta_0(N) = Z(N)$, then a 0-cochain assigns to each $([\ ]; U_i)$ a map $\alpha([\ ]; U_i) \in \text{map}(W_i, T)$. Of course $(\delta d)(\alpha; U_i, U_j)$ $= \alpha_{\#}\big(d([\ ]; U_j)\big)\big(d([\ ]; U_i)\big)^{-1}$ in $\text{map}(W_i \cap W_j, T)$. Putting $\zeta_i(w)$ equal to the value of $d([\ ]; U_i)$ at $w \in W_i$ we see that $\alpha_*\big(\zeta_j(w\alpha)\big)\zeta_i(w)^{-1}$ corresponds to δd. Thus, as asserted, $H^1_{\mathcal{U}}(N;\mathcal{J})$ is the group of equivalence classes of holomorphic co-ordinate systems with operators.

It is not difficult to introduce $H^*(N;\mathcal{J}) = \text{dirlim}\, H^*_{\mathcal{U}}(N;\mathcal{J})$ taken over open coverings of V. The usual considerations apply here; that is, if $\mathcal{U}'$ refines $\mathcal{U}$ then each refining function $r: \mathcal{U}' \to \mathcal{U}$ with $U' \subset r(U)$ defines a covariant functor (simplicial map) $N(\mathcal{U}') \to N(\mathcal{U})$. All refining functions define simplicial maps of the same simplicial homotopy type, thus if $\mathcal{U}' < \mathcal{U}$ then $H^*_{\mathcal{U}}(N;\mathcal{J}) \to H^*_{\mathcal{U}'}(N;\mathcal{J})$ is well defined.

The reader may feel that there is some difference between the definition of $H^*_{\mathcal{U}}(N;\mathcal{J})$ above and what he might expect in the light of the definition of $H^*(N;\mathcal{J})$ given in section 2. The difference is that of forming a cartesian product of simplicial complexes versus the formation of the tensor product of two chain complexes. Thus, following Kodaira [23], we might have used the functor

$$\sigma \to \beta(N) \otimes C(\sigma)$$

where $\big(\beta(N) \otimes C(\sigma)\big)_p = \sum_{j+1=p} \beta_j(N) \otimes C_i(\sigma)$. This would yield $C^*_{\mathcal{U}}(N;\mathcal{J})$ as the double complex $\sum C^{i,j}(N;\mathcal{J})$ equipped with a coboundary operator of the form $\delta = \delta' + \delta''$. From the Eilenberg-Zilber theorem [25, p. 238] we can see that these two definitions of $H^*_{\mathcal{U}}(N;\mathcal{J})$ agree. Indeed, if for each $\sigma \in N(\mathcal{U})$, $f_\sigma : \beta(N) \otimes C(\sigma) \to \beta(N) \times C(\sigma)$ is the Alexander-Whitney map, and $g_\sigma : \beta(N) \times C(\sigma) \to \beta(N) \otimes C(\sigma)$ is the map defined using shuffles then both $\{f_\sigma\}$ and $\{g_\sigma\}$ are natural transformations between functors. Further $\{f_\sigma g_\sigma\}$ and $\{g_\sigma f_\sigma\}$ are chain homotopic to the identity and in each case the chain homotopy is given by a pair of natural transformations of the functor into itself.

We shall mention briefly the role of $e_*: H^1(N; \mathcal{O}(w)^k) \to H^1(N; \mathcal{J})$. Suppose $\tau \in H^1(N; \mathcal{J})$ is an element for which $\delta(\tau) \in \mathcal{B}_\Phi$. Let $\{m_{i,j}(w, \alpha)\}$ be a holomorphic coordinate system with operators representing τ. Suppose $v \in H^1(N; \mathcal{O}(W)^k) \simeq H^1(V; h_c^0)$, then for the same covering $\mathcal{U}$ of V we may suppose that $\{M_{i,j}(w, \alpha)\}$ is a holomorphic coordinate system with operators with values in C^k which represents v. For every complex $\lambda \in C$, $\{\lambda M_{i,j}(w, \alpha)\}$ represents λv. Let us then define a <u>parameterized holomorphic coordinate system with operators with values in</u> T

$$m_{i,j}(\lambda, w, \alpha) = \exp\left(\lambda M_{i,j}(w, \alpha)\right) \cdot m_{i,j}(w, \alpha) .$$

When $\lambda = 0$ this represents τ and when $\lambda = 1$ it represents $\tau + e_*(v)$. We form $(T, D, N) \to (C \times W, N)$ by introducing into the disjoint union $\bigcup (T \times C \times W_i \times \{i\})$ the equivalence relation $(t', \lambda', w', j) \sim (t, \lambda, w, i)$ if and only if $\lambda' = \lambda$, $w = w'$ and $t' m_{i,j}(\lambda, w, e) = t$. The projection map is $((t, \lambda, w, i)) \to (\lambda, w)$. The action of N on the C factor is taken to be trivial, thus, if (D, N) is defined by

$$((t, \lambda, w, i))\alpha = \left(\left(\alpha_*^{-1}(t) m_{i,i}(\lambda, w\alpha, \alpha^{-1}), \lambda, w\alpha, i\right)\right)$$

the projection map is N-equivariant. Now $\delta\left(\tau + e_*(\lambda v)\right) = \delta(\tau) \in \mathcal{B}_\Phi$ so that N acts freely on D. We introduce the holomorphic Seifert fibration $D/N \to C \times V$ by $[t, \lambda, w, i] \to (\lambda, \nu(w))$. This is what we refer to in saying $M_\tau \to V$ <u>can be deformed into</u> $M_{\tau + e_*(v)} \to V$.

5. $\dim_c W = 1$.

To illustrate all the preceding general considerations let us pause now to consider this case. We may regard (W, N) as a ramified covering of the non-singular Riemann surface $V = W/N$. In this section we shall also assume that the restriction of $\Phi: N \to \mathrm{Aut}(T)$ to each isotropy subgroup is trivial so that by (3.6) Φ is a homomorphism of $\pi_1(V)$ into Aut(T). According to (2.4), $h_c^0 \to V$ is then a locally free sheaf of rank k.

(5.1) <u>Theorem</u>: <u>If</u> $\dim_c W = 1$ <u>then</u> $H^1(N; \mathcal{J}) \to H^2(N; \mathfrak{Z}^{2k})$ <u>is an epimorphism and</u>

$$0 \to H^2(V; h^0) \to H^2(N; \mathfrak{Z}^{2k}) \to H^0(V; h^2) \to 0$$

<u>is a short exact sequence</u>. <u>If</u> V <u>is an open Riemann surface then</u> $H^1(N; \mathfrak{J}) \simeq H^2(N; \mathfrak{Z}^{2k})$. <u>If</u> W <u>is open and simply connected then</u> $H^2_\Phi(N; Z^{2k}) \simeq H^2(N; \mathfrak{Z}^{2k})$.

<u>Proof</u>: Since $h^0_c \to V$ is coherent and $\dim_c V = 1$ it follows that $H^2(V, h^0_c) = 0$, thus every Bieberbach class has a holomorphic realization. Referring to (3.5) and the fact that $\dim_R V = 2$ we also have the short exact sequence

$$0 \to H^2(V; h^0) \to H^2(N; \mathfrak{Z}^{2k}) \to H^0(V; h^2) \to 0 .$$

Any open Riemann surface is a Stein manifold so that if V is also open then $H^1(V; h^0_c) = 0$. Finally, an open simply connected Riemann surface is of course contractible so that by (3.9) $H^2_\Phi(N; Z^{2k}) \simeq H^2(N; \mathfrak{Z}^{2k})$.

(5.2) <u>Corollary</u>: <u>If</u> (W, N) <u>is a properly discontinuous group of holomorphic transformations on an open simply connected Riemann surface then to every torsionless extension</u> $0 \to Z^{2k} \to \pi \to N \to 1$ <u>compatible with</u> Φ <u>there is associated an aspherical complex manifold</u> M, <u>with fundamental group</u> π, <u>together with a holomorphic local action of</u> T, $\psi: M \to V$.

If V is compact, then M is also closed. The properly discontinuous transformation groups on open simply connected Riemann surfaces have long been known, of course. Using (3.11) we can always replace (W, N) by (W^*, N^*) so that unless W is already closed and simply connected we can always apply (5.2).

Let us recall from [13, sec. 10] how N acts on an open simply connected W. Every non-trivial finite subgroup of N is cyclic and has a unique fixed point. Every non-trivial finite subgroup lies in a unique maximal finite subgroup, the isotropy group of its fixed point. Every non-trivial isotropy subgroup is its own normalizer. Let $S \subset W$ be the set of elements with non-trivial isotropy subgroup. Then S is an N-invariant subset and $S/N \subset V$ is a discrete closed subset in natural 1-1 correspondence with the conjugacy classes of the non-trivial isotropy subgroups. The sheaf $h^2 \to V$ is trivial on the complement of S/N. Since $h^2_{\nu(w)} \simeq H^2(N_w; Z^{2k}) \simeq \mathrm{Hom}(N_w, T)$ we see that a section of $h^2 \to V$ is given by choosing a homomorphism into T for a representative from each conjugacy class of the non-trivial isotropy subgroups. Since $H^2_\Phi(N; Z^{2k}) \to H^0(V; h^2)$ is an epimorphism we can state

(5.3) Theorem: Let (W, N) be a properly discontinuous group of holomorphic transformations on an open simply connected Riemann surface. Modulo the sum with an element in the image of $0 \to H^2(V; h_c^0) \to H_\Phi^2(N; Z^{2k})$ a Bieberbach class is uniquely determined by choosing an isomorphic embedding into T for a representative from each conjugacy class of the non-trivial isotropy subgroups.

6. The subset A_Φ

Under the quite rigid hypothesis that W be 2-connected we observed in (3.9) that A_Φ may be identified with $\mathfrak{B}_\Phi$. Under a weaker hypothesis a geometric interpretation can be given separately for A_Φ. To $\mathrm{map}(W, C^k)$, the linear space of all holomorphic maps of W into C^k, we assign the $Z(N)$-module structure given by $\alpha_\#(f)(w) = \alpha_*\big(f(w\alpha)\big)$.

(6.1) Lemma: If W is a Stein manifold, then $H_\Phi^*\big(N; \mathrm{map}(W, C)\big) \simeq H^*(V; h_c^0)$.

Proof: This is an analogue to (3.9). In (2.3) we showed that $H^*(V; h_c^0) \simeq H^*\big(N; \mathcal{O}(W)^k\big)$. There is also the "E-spectral for $H^*\big(N; \mathcal{O}(W)^k\big)$ with "$E_2^{i,j} \simeq H^i\Big(N; H^j\big(W, \mathcal{O}(W)^k\big)\Big)$. Since W is a Stein manifold, however, $H^j\big(W; \mathcal{O}(W)^k\big) = 0$ for all $j > 0$. Thus we have

$$H_\Phi^*\big(W; \mathrm{map}(W, C^k)\big) \simeq H^*\big(N; \mathcal{O}(W)^k\big) \simeq H^*(V; h_c^0) \quad .$$

Now let us add the assumption that W is simply connected. Then we have a short exact sequence of $\mathbf{Z}(W)$-modules $0 \to Z^{2k} \to \mathrm{map}(W, C^k) \to \mathrm{map}(W, T) \to 0$. This yields another exact cohomology exact triangle and a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & H_\Phi^1(N; Z^{2k}) \arrow[dr] \arrow[dd, "\simeq"] & & H_\Phi^1\big(N; \mathrm{map}(W, T)\big) \arrow[r] \arrow[dd] & H_\Phi^2(N; Z^{2k}) \arrow[dr] \arrow[dd, "\sim"] & & {} \\
& & H^1(V; h_c^0) \arrow[ur] \arrow[dr] & & & H^2(V; h_c^0) \arrow[ur] \arrow[dr] & \\
{} \arrow[r] & H^1(N; \mathfrak{Z}^{2k}) \arrow[ur] & & H^1(N; \mathfrak{T}) \arrow[r] & H^2(N; \mathfrak{Z}^{2k}) \arrow[ur] & & {}
\end{tikzcd}$$

Appeal to (3.8). Using that, and the above commutative diagram, we may infer

(6.2) Theorem: If W is a simply connected Stein manifold then

$$H^1_\Phi(N; \mathrm{map}(W,T)) \to H^1(N; \mathcal{J})$$

<u>is a monomorphism</u>. <u>If</u> $\tau \in H^1(N;\mathcal{J})$ <u>then</u> τ <u>lies in the image of this monomorphism if and only if</u> $\delta(\tau)$ <u>lies in the kernel of the edge homomorphism</u>

$$H^2(N; \mathfrak{z}^{2k}) \to H^0(N; H^2(W; Z^{2k})) \quad .$$

The result is interpreted as follows. To $\tau \in H^1(N;\mathcal{J})$ there is associated a holomorphic left principal T-bundle with operators $(T, B, N) \to (W, N)$. As a topological T-bundle it has a characteristic class in $H^2(W; Z^{2k})$. Because the bundle has operators covering (W, N) this characteristic class lies in the subgroup of elements of $H^2(W; Z^{2k})$ which are fixed under the action of N; that is, in $H^0(N; H^2(W;Z^{2k}))$. Thus the assumption that $\delta(\tau)$ lies in the kernel of $H^2(N;\mathfrak{z}^{2k}) \to H^0(N; H^2(W; Z^{2k}))$ means that the T-bundle is topologically trivial. The assertion of (6.2) is that it is also holomorphically trivial, which would be expected since W is a Stein manifold.

There is a simplification associated to the construction of Seifert fibrations from the elements of $H^1_\Phi(N; \mathrm{map}(W,T))$ for which $\delta(\tau) \in A_\Phi$. A 1-cocycle in $Z^1_\Phi(N; \mathrm{map}(W,T))$ may be regarded as a holomorphic $m: W \times N \to T$ satisfying $m(w, \alpha, \beta) = m(w, \alpha)\alpha\left(m(w\alpha, \beta)\right)$. The action $(T \times W, N)$ is $(t, w)\alpha = \left(\alpha_*^{-1}(t) m(w\alpha, \alpha^{-1}), w\alpha\right)$. Since $\delta(\tau) \in A_\Phi$ we put $M_\tau = (T \times W)/N$ and $\psi((t,w)) = \nu(w)$. The fundamental group of M_τ is given by the extension $0 \to Z^{2k} \to \pi \to N \to 1$ corresponding to $\delta(\tau) \in A_\Phi \subset H^2_\Phi(N; Z^{2k})$.

(6.3) <u>Lemma</u>: <u>If</u> $A_\Phi \neq \emptyset$, N <u>finitely generated, there is a normal subgroup</u> $L \subset N$, <u>of finite index</u>, <u>such that for every isotropy subgroup</u>

(i) $N_w \cap L$ <u>is isomorphic to a subgroup of</u> T

(ii) $\Phi | N_w \cap L \to \mathrm{Aut}(T)$ <u>is trivial</u>.

<u>Proof</u>: Let $K \subset N$ denote the kernel of Φ and let

$$0 \to Z^{2k} \to \pi \to N \to .$$

represent an element of $A_\Phi \subset H^2_\Phi(N; Z^{2k})$. There is induced a central extension

$$0 \to Z^{2k} \to G \to K \to 1$$

which represents an element of $A \subset H^2(K; Z^{2k})$. Now the isotropy subgroups of (W, K) are

$K_W = K \cap N_W$. We can apply (3.5) to (W, K) so that each K_W is isomorphic to a subgroup of T. But $\Phi: N/K \to \mathrm{Aut}(T) \subset GL(k, C)$ surely embeds N/K into $GL(k, C)$ as a finitely generated discrete subgroup. According to a theorem of E. Cartan [5], there must be a normal subgroup $H \subset N/K$ which is torsionless and has finite index. Then we take $K \subset L \subset N$ so that $L/K = H$. Since L/K is torsionless it must follow that $N_W \cap L = N_W \cap K = K_W$ for each one of the isotropy subgroups. This establishes the lemma.

We point this out frankly because we are not clear on the interpretation of the Seifert fiberings which arise when the restrictions $\Phi | N_W \to \mathrm{Aut}(T)$ are not all trivial. We have said the following. Suppose $\tau \in H^1_\Phi(N; Z^{2k})$ has $\delta(\tau) \in A_\Phi$, then $M_\tau \to V$ is constructed. There is a holomorphic local action of T,

$$\begin{array}{ccc} M'_\tau & \longrightarrow & V' = W/L \\ \downarrow & & \downarrow \\ M_\tau & \longrightarrow & V = W/N \end{array}$$

which is a finite covering of the original Seifert fibration. We simply observe that in $(T \times W, N)$ the normal subgroup L of (6.3) is also operating freely so we put $M'_\tau = (T \times W)/L$, $V' = W/L$, and the finite N/L still acts freely on M'_τ with quotient M_τ. The Seifert fibration $M'_\tau \to V'$ is a local holomorphic action of T since every restriction $\Phi | L_W \to \mathrm{Aut}(T)$ is trivial. An immediate corollary of (6.3) is

> (6.4) Lemma: Suppose N is finitely generated and that $\Phi | N_W \to \mathrm{Aut}(T)$ is faithful on every isotropy subgroup then there is a normal subgroup $L \subset N$, of finite index, which acts freely on W.

We cannot assert in general that L lies in the kernel of Φ, so we may not be able to apply (3.10). It is only under the hypothesis of (6.4) that we can assert that each non-trivial isotropy subgroup N_W is exactly the holonomy group of the singular fibre T/N_W of a Seifert fibration constructd from an element in $H^1_\Phi(N; \mathrm{map}(W, T))$. In this case, however, the Seifert fibration will be finitely covered by a local holomorphic action $M'_\tau \to V'$ without singular fibres and with local holomorphic sections.

The lemma 6.3 may be formulated also in terms of $H^1(N; \mathcal{J})$: If $\tau \in H^1(N, \mathcal{J})$ is such that $\delta(\tau) \in B_\Phi$, then we may obtain the same conclusion as in 6.3 by essentially the same arguments. In particular, we have the

(6.5) Corollary: If τ represents a local action, then M_τ has a (not necessarily finite) covering $M_{'\tau}$, with covering transformations N/K, where K denotes the kernel of Φ. The holomorphic Seifert fibre space $M_{'\tau}$ arises from a holomorphic action and each of its orbits projects homeomorphically onto $p^{-1}(v) \subset M_\tau$ for some $v \in W/N$. Furthermore, the holomorphic covering $W/K \longrightarrow W/N$ is unbranched.

Our experience suggests that working with

$$\begin{tikzcd} H^*_\Phi(N; Z^{2k}) \arrow[rr] & & H^*(V; h^0_c) \arrow[dl] \\ & H^*_\Phi(N; \mathrm{map}(W, T)) \arrow[ul] & \end{tikzcd}$$

is preferable to the general exact cohomology triangle which we studied prior to this section.

7. The Smooth Case

We may also consider a properly discontinuous group of diffeomorphisms (W, N). We would then consider a real k-torus $0 \to Z^k \to R^k \to T \to 0$ and a homomorphism $\Phi: N \to GL(k, Z)$. Using smooth maps we can define the sheaf with operators $\mathcal{J}_R \to W$, and of course $\mathfrak{Z}^k \to W$ is defined as before. The basic change is

(7.1) Theorem: For $j > 0$ in the smooth case

$$\delta: H^j(N; \mathcal{J}_R) \simeq H^{j+1}(N; \mathfrak{Z}^k) \quad .$$

Proof: By analogy with $\mathcal{O}(W)^k \to W$ there is the sheaf with operators $\mathcal{R}(W)^k \to W$, the sheaf of germs of smooth maps on W into R^k. Just as in (2.3) we show that the edge homomorphism $H^*(N; \mathcal{R}(W)^k) \to H^*(V; h^0_R)$ is an isomorphism. In this case, however, $h^0_R \to V$ is a fine sheaf. Recall that the sections of $h^0_R \to V$ over V are the smooth maps $f: W \to R^k$ which satisfy the identity $\alpha_*(f(w\alpha)) = f(w)$ for all $\alpha \in N$. If $\mathcal{U}$ is a locally finite open covering of V then there is a partition of unity $\{\epsilon_i\}$ subordinate to $\mathcal{U}$ with the property that the composite maps $s_i = \epsilon_i \circ \nu: W \to R$ are all smooth. Now using the fact that α_* is real linear and that $s_i(w\alpha) = s_i(w)$ we see that $g(w) \quad s_i(w)f(w)$ is a global section of

$h^0_R \to V$ with support in U_i. Of course $\sum_i s_i(w) f(w) = f(w)$ and thus $h^0_R \to V$ is a fine sheaf.

Using this, the "E-spectral sequence, and the fact that $\mathcal{R}(W)^k$ is fine also, it is immediately seen that $H^j_\Phi(N; \mathrm{Map}(W, R^k)) = 0$ for all $j > 0$. Here $\mathrm{Map}(W, \cdot)$ is the real linear space of smooth maps.

(7.2) Corollary: *If (W, N) is a properly discontinuous group of diffeomorphisms on a simply connected manifold then there is a commutative diagram*

$$\begin{tikzcd} H^1_\Phi(N; \mathrm{Map}(W, T^k)) \arrow[r, phantom, "\simeq"] \arrow[d] & H^2_\Phi(N; Z^k) \arrow[d, "(\text{monic})"] \\ H^1(N; \mathcal{J}_R) \arrow[r, phantom, "\simeq"] & H^2(N; \mathfrak{Z}^k) \end{tikzcd}$$

Every statement in section 3 remains valid if $2k$ is replaced by k. We would like to proceed now to the case Φ is trivial. Then to every Bieberbach class we assoicate a smooth action (T, M) of a k-torus on a manifold having at most finite isotropy groups. We ask just which smooth actions can be obtained by this process. Suppose we are given a (T, M). At each $x \in M$ there is $f^x : T \to M$ given by $f^x(t) = tx$. This induces $f^x_* : \pi_1(T) \to \pi_1(M, x)$, the image of which is central. We define a canonical homomorphism $\eta_x : T_x \to \pi_1(M, x)/\mathrm{im}(f^x_*)$ on the isotropy subgroup into the quotient [13 sec. 4]. This is done as follows. If t leaves x fixed choose any path, $p(\tau)$, in T with $p(0)$, $p(1) = t$. Then $p(\tau) \cdot x$ is a closed loop in M at x and in the quotient group it represents $\eta_x(t)$.

(7.3) Theorem: *Let (T, M) be a smooth action of a k-torus on a manifold. There is a properly discontinuous group of diffeomorphisms (W, N) on a simply connected manifold and a Bieberbach class in $H^2(N; \mathfrak{Z}^k)$ which, up to an equivariant diffeomorphism, yields (T, M) if and only if η_x is a monomorphism at every point of M.*

If (T, M) is a holomorphic action of a complex toral group then (W, N) is a properly discontinuous holomorphic action. The holomorphic action arises from a holomorphic coordinate system with operators $(T, B, N) \to (W, N)$ which is determined by a class $\tau \in H^1(N, \mathcal{Y})$ for which $\delta(\tau)$ is a Bieberbach class.

Proof: Suppose first that all η_x are monomorphisms. Choose a point $x_0 \in M$. There is a covering action $(T, B, N) \to (T, M)$ where B is the covering space corresponding to the

subgroup $\operatorname{im}(f_*^{x_0}) \subset \pi_1(M, x_0)$; $N = \pi_1(M, x_0)/\operatorname{im}(f_*^{x_0})$ is the group of covering transformations and the actions of T and N commute [13]. Combining (4.16) and (4.7) of [13] we find that (T, B) is a principal action if and only if η_x is a monomorphism at each point. Now $f_*^{b_0}: \pi_1(T) \to \pi_1(B, b_0) = \operatorname{im}(f_*^{x_0})$, thus $B/T = W$ is simply connected and we receive a properly discontinuous action (W, N) with $W/N = V = M/T$.

We must now associate a smooth co-ordinate system with operators to $(T, B, N) \to (W, N)$. We assert that there is an open covering $\mathcal{U}$ of V such that over every $W_i = \nu^{-1}(U_i)$ there is a local section $s_i: W_i \to B$. On $(W_i \cap W_j) \times N_1$ where $W_i \cap W_j \neq \emptyset$, we put

$$\left(s_i(w)\right)\alpha = m_{i,j}(w, \alpha)\left(s_j(w\alpha)\right) .$$

This defines the cooordinate system with operators and the $\tau \in H^1(N; \mathfrak{J}_R)$. The Bieberbach class is $\delta(\tau)$.

Now we must go the other way. Thus (W, N) is a properly discontinuous group of diffeomorphisms on a simply connected manifold and (T, M) is derived from some Bieberbach class. We have then a diagram

$$\begin{tikzcd}
 & (T, B, N) \arrow[dl] \arrow[dr] & \\
(T, M) & & (W, N)
\end{tikzcd}$$

and choosing a base point $b_0 \in B$

$$\begin{tikzcd}
 & \pi_1(B, b_0) \arrow[d] \\
\pi_1(T) \arrow[ur, dashed, "f_*^{b_0}"] \arrow[r, "f_*^{x_0}"'] & \pi_1(M, x_0)
\end{tikzcd}$$

Since W is simply connected, $f_*^{b_0}: \pi_1(T) \to \pi_1(B, b_0)$ must be an epimorphism and so (T, B, N) is the covering action associated to $\operatorname{im}(f_*^{x_0})$. Thus $N \simeq \pi_1(M, x_0)/\operatorname{im}(f_*^{x_0})$, and since (T, B) is principal it must follow that $\eta_x : T_x \to \pi_1(M, x)/\operatorname{im}(f_*^{x})$ is a monomorphism at every point. By (3.11) there is no loss of generality in assuming that W is simply connected so that we have described all smooth actions of T which can be constructed from a Bieberbach class.

In the holomorphic case the covering action $(T, B, N) \to (T, M)$ of the complex torus is also holomorphic. Since Holmann has shown the existence of holomorphic slices [20], the

principal bundle with operators (T, B, N) is a principal holomorphic bundle over $B/T = W$, and N operates holomorphically on W.

We have in a canonical manner associated to a smooth (T, M) satisfying the hypothesis of (7.2) a properly discontinuous group of diffeomorphisms (W, N) and a Beiberbach class in $H^2(N; \mathfrak{Z}^k)$. Let us say that this is the <u>characteristic class</u> of (T, M) and that (T, M) has a characteristic class.

(7.4) <u>Theorem</u>: <u>Let</u> (T, M) <u>be a smooth action and suppose that at some point</u> $x \in M$, $f^x_*: \pi(T) \to \pi(M, x)$ <u>is a monomorphism.</u> <u>Then</u> (T, M) <u>has a characteristic class which lies in the image of</u>

$$0 \to H^2(N; Z^k) \to H^2(N; \mathfrak{Z}^k) .$$

<u>Proof</u>: This is an injective action as defined in [14]. It was shown that if (T, B, N) is the covering action corresponding to $\mathrm{im}(f^x_*)$ then not only is (T, B) a principal action but the principal bundle $B \to B/T = W$ admits a global cross-section. Thus the characteristic class lies in the kernel of $H^2(N; \mathfrak{Z}^k) \to H^0(N; H^2(W; Z^k))$ and hence by (3.7) in the image as asserted.

If (T, M) is assumed to be holomorphic it is of course also smooth. However, $B \to B/T = W$ which admits a smooth cross-section may not admit a holomorphic one. Let $\tau \in H^1(N, \mathcal{J})$ which yields (T, M), then there is a class $b \in H^2(N; \mathrm{map}(W; Z^{2k}))$ whose image is $\delta(\tau) \in H^2(N, \mathfrak{Z}^{2k})$, if (T, M) is injective. From the exact sequence $H^1(N; \mathrm{map}(W, T)) \to H^2(N; \mathrm{map}(W, Z^{2k})) \xrightarrow{j} H^2(N; \mathrm{map}(W, C^k))$, $j(b) = 0$, if and only if the principal holomorphic bundle (T, B, N) with operators representing τ is holomorphically trivial. As seen before, this would be the case if W is Stein.

We would like to illustrate some of the foregoing material by two examples. In the first example we take a finite group N and we let it operate trivially on a point w. To be a Bieberbach class in $H^2(N, \mathfrak{Z}^k)$ $\left(= H^2_\Phi(N; Z^k) \text{ since } W \text{ is contractible}\right)$ the group extension $Z^k \to \pi \to N$ must be torsion free. The Seifert fibering consists of a single fibre T/N over a single point. The manifold T/N is of course a flat manifold and $\Phi: N \to \mathrm{Aut}\, T$ may as well be assumed faithful. (If not, then T/K, K the kernel of ϕ, is once again a torus, since $\chi_w: K = K_w \to T$ must be a monomorphism. We may then consider N/K instead.) Clearly <u>all flat manifolds with holonomy</u> N (see 6.4) <u>may be constructed this way</u>. Furthermore the Bieberbach classes of $H^2_\Phi(N; Z^k)$ can be used to determine the flat manifolds up to affine diffeomorphisms. This program is carried out in the work of <u>Charlap</u> [11]. It can also be

carried out in our general framework. What is needed is a suitable generalization of Theorem 6.6 of [13]. This generalization takes account of automorphisms of T, compatible homomorphisms ϕ as well as what is discussed in [13]. In the torsion free extension $0 \to Z^k \to \pi \to N \to 1$, the subgroup Z^k is a characteristic subgroup of π. This then can be used to show that if M_τ and $M_{\tau'}$ have isomorphic fundamental groups there they are (affinely) diffeomorphic. We do not pursue the details here.

The next examples are spaces of constant positive curvature. Let us consider M^{2n+1} as obtained from a free linear action on S^{2n+1} by a finite subgroup F of U(n). The center U(1) of diagonal matrices commutes with F and so induces an action of $U(1)/U(1) \cap F$ on M with only finite stability groups. We now assume that n is the smallest integer for which F has a faithful free representation. The center K of F is precisely $F \cap U(1)$. By studying a certain eigenvalue problem Dost Khan [22] has shown that $f^x : \pi_1(U(1)/K, 1) \longrightarrow \pi_1(M, x)$ has image precisely the center K. All the stability groups are embeded by $\eta_x : (U(1)/K)_x \longrightarrow \pi_1(M, x)/K$ monomorphically. In fact, each singular orbit is <u>isolated</u>. If we form $S^{2n+1}/U(1) = CP_n$ we have induced the action of F/K. This (holomorphic) action has only isolated points at which the stability group is not trivial. The stability groups are all cyclic and are isomorphic to the corresponding stability groups of $(U(1)/K, M)$. We may form $H^2(F/K; \mathfrak{Z})$. The smooth Seifert fibering $(U(1)/K, S^{2n+1}/F) = M_\tau$ arises from a Bieberbach class $\delta(\tau) \in H^2(F/K; \mathfrak{Z})$. The smooth bundle with operators $(T, B, X) \to (W, X)$ is just the lens space $(U(1)/K, S^{2n+1}/K, F/K)$ over $(CP_n, F/K)$. If $n > 1$, then $CP_n/(F/K)$ is an analytic space with singularities (in fact a variety).

If one selects $n = 1$, we may construct all closed 3-manifolds of constant positive curvature whose fundamental groups are non-abelian as Seifert fiberings over the 2-sphere. The number of singular orbits in this case is always precisely three. The non-abelian groups F appearing have faithful representations of F/K in SO(3) and represent standard finite groups of isometries of the complex projective line. By the classification of all circle actions on 3-manifolds and the classification of 3-dimensional spherical space forms, every Bieberbach class of $H^2(N; \mathfrak{Z})$, where N operates on S^2, $\phi : N \to \mathrm{Aut}(S^1)$ is trivial, and the resulting three manifold M is not a lens space, must arise this way. If instead of taking a principal S^1 bundle we were to take a principal (holomorphic) C^*-bundle we would obtain an open complex 4-manifold with C^*-action diffeomorphic to $R^1 \times M^3_\tau$. By taking C^*/Z, a complex torus, there is induced an analytic action on $(T^2, S^1 \times M^3_\tau)$ which is represented by a Bieberbach class in $H^2(N; \mathfrak{Z})$. Since M^3_τ is a rational cohomology 3-sphere, the holomorphic Seifert fibering $S^1 \times M^3_\tau$ admits no Kahlerian structure and hence can not be algebraic. The fundamental group is $Z \times (\text{torsion})$.

Let us look at a very specific example. We choose F/K to be the icosahedral group (isomorphic to the alternating group on five elements). The group F is the central extension of $Z_2 = K$ by $I = F/K$. The icosahedral group acts on CP_1 by isometries. From the 'E and "E spectral sequences (and using $H^1(S^2/I; Z) = H^1(S^2; Z) = 0$) we have from the exact sequences of terms of low degree:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^2(I; H^0(S^2, Z)) \arrow[d] & & \\
0 \arrow[r] & H^2(S^2/I; Z) \arrow[r] & H^2(I; \mathfrak{Z}) \arrow[r] \arrow[d] & H^0(S^2/I; h^2) \arrow[r] & 0 \\
 & & H^0(I; H^2(S^2; Z)) \arrow[d] & & \\
 & & H^3(I; Z) & &
\end{tikzcd}$$

Substituting we have:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & 0 \arrow[d] & & \\
0 \arrow[r] & Z \arrow[r] & H^2(I; \mathfrak{Z}) \arrow[r, "\nu"] \arrow[d, "i"] & Z_2 \oplus Z_3 \oplus Z_5 \arrow[r] & 0 \\
 & & Z \arrow[d] & & \\
 & & Z_2 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

Thus $H^2(I; \mathfrak{Z}) \cong Z$ but not every element represents a Bieberbach class. If $\tau \in B$, then $\nu(\tau)$ must generate $Z_2 \oplus Z_3 \oplus Z_5 = Z_{30}$. The homomorphism i sends the generator onto the class 2 which is the characteristic class of the circle bundle with I as operators over the 2-sphere. That is the total space is real projective 3-space. We may generate other Beiber-bach classes and bundles with operators by adding to τ the image of elements of $H^2(S^2/I; Z) \approx Z$ or by choosing another element whose image under ν is a generator of Z_{30}. The characteristic class of the circle bundle to be taken over S^2 changes with these different

choices. For our generator τ, the Seifert fibre space M_τ has $\pi_1(M_\tau) \cong$ binary icosahedral group, F, and M_τ is the well known Poincaré sphere. It is a homogeneous space and in terms of our description at the beginning of this section it originates in $(U(1), SU(2), F)$. The coordinate bundle with operators yields

$$\begin{array}{ccc} (U(1)/Z_2, SU(2)/Z_2 = SO(3), F/Z_2 = I) & \xrightarrow{/U(1)/Z_2} & (S^2, I) \\ \big\downarrow /I & & \big\downarrow /I \\ (U(1)/Z_2, SO(3)/I = M_\tau) & \xrightarrow{/U(1)/Z_2} & (S^2/I = S^2) \end{array}$$

The other possibilities have $\pi_1(M) \cong Z_n \times F$ where n is relatively prime to 30. They may be constructed from co-ordinate bundles with operators,

$$\left(U(1)/Z_2 \times Z_n,\ SU(2)/Z_2 \times Z_n = \text{lens space},\ F \times Z_n/Z_2 \times Z_n = I\right)$$

or by taking

$$\left(U(1)/Z_2, SO(3)/I\right) \xrightarrow{/Z_n} \left(U(1)/Z_2 \times Z_n, SO(3)/I/Z_n\right),$$

where $Z_n \subseteq U(1)$.

As we shall see in section twelve, every real 4-manifold arising from a Bieberbach class of $H^2(F/K; \mathfrak{Z}^2)$ where F is a finite non-abelian group having a free linear representation in U(2) admits a complex structure. We shall see that they fiber smoothly over the circle but not the torus. Hence, their first betti number will be odd and consequently are never Kähler manifolds. We do not know if these 4-manifolds are always products of the circle with a closed 3-manifold of constant positive curvature.

Finally we come to the matter of fibrations in the smooth case. We assume that $H_1(M; Z)$ is finitely generated.

(7.5) <u>Theorem: The following are equivalent for a smooth action</u> (T, M):

(i) <u>there is a smooth fibre map</u> $g: M \to T$, <u>satisfying</u> $g(tx) = t^n g(x)$ <u>for some</u> $n > 0$, <u>with a finite abelian structure group;</u>

(ii) $f_*^x: H_1(T; Z) \to H_1(M; Z)$ <u>is a monomorphism;</u>

(iii) (T, M) <u>has a characteristic Bieberbach class of finite order.</u>

<u>Proof:</u> That (i) and (ii) are equivalent was proved in [14]. It was also shown, in view of (7.4), that (i) implies (iii) (actually in [14] the characteristic class was considered to be an element of $H^2(N; Z^k)$). We postpone the proof that (iii) implies (i) to the next section where a similar argument in the analytic case is presented.

(7.6) <u>Corollary: Let (W, N) be a properly discontinuous group of diffeomorphisms on a simply connected manifold. There is a Bieberbach class of finite order in $H^2(N; \mathfrak{Z}^k)$ if and only if there is a normal subgroup $K \subset N$, acting freely on W, for which the quotient group N/K is finite and a homomorphic image</u> of Z^k.

<u>Proof</u>: Suppose that there is a Bieberbach class of finite order. Then this is the characteristic class of a (T, M) and by (7.5) there is a smooth map $g: M \to T$ which for some $n > 0$ satisfies $g(tx) \equiv t^n g(x)$. We choose a base point $x_0 \in M$ with $g(x_0) = e \in T$, then the composition $gf^{x_0}: T \to T$ is $t \to t^n$. We take $K \subset \pi_1(M, x_0)$ to be the kernel of $g_*: \pi_1(M, x_0) \to \pi_1(T)$. Now $\operatorname{im}(f_*^{x_0}) \cap K = \{1\}$ so that K embeds isomorphically into $\pi_1(M, x_0)/\operatorname{im}(f_*^{x_0}) - N$. We denote its image by K also. The splitting theorem asserts that K acts freely on W [14]. In addition, $N/K \simeq \pi_1(M, x_0)/(\operatorname{im}(f_*^{x_0}) \cdot K)$ is isomorphic to a subgroup of $(Z_n)^k$.

To prove the converse we assume that the subgroup K exists. Then there must be a short exact sequence

$$0 \to Z^k \to Z^k \to N/K \to 0 .$$

But this is a torsionless central extension of Z^k by N/K and so represents a Bieberbach class of finite order in the image of

$$H^2(N/K; Z^k) \to H^2(N/K; \mathfrak{Z}^k) \to H^2(N; \mathfrak{Z}^k) .$$

Let us return to the holomorphic case for a moment to observe

(7.7) <u>Corollary: Let (W, N) be a properly discontinuous group of holomorphic transformations on a complex manifold for which W/N = V is compact. If for some</u> $k > 0$ <u>there is a</u> $\tau \in H^1(N; \mathcal{J})$ <u>for which</u>

(i) $\delta(\tau) \in \mathcal{B} \subset H^2(N; \mathfrak{Z}^{2k})$

(ii) M_τ is a projective algebraic variety

then there is a normal subgroup $K \subset N$ acting freely on W for which N/K is a finite abelian group.

Proof: We have a holomorphic action (T, M_τ) on a projective variety with only finite isotropy subgroups. It follows trivially from Matsushima's results [26] that $f_*^{x_0}: H_1(T; Z) \to H_1(M_\tau; Z)$ is a monomorphism.

It should be pointed out that there are many holomorphic (W, N) where $\dim_c W = 1$ and $W/N = V$ is an algebraic curve, but for no value of $k > 0$ does $\mathcal{B} \subset H^2(N; \mathfrak{Z}^{2k})$ contain an element of finite order. We refer the reader to [15] for a general treatment of this phenomenon. Also by using 12.6, the Corollary 7.7 can be considerably extended. We shall refer to 7.7 again in the next section.

8. Holomorphic Actions of T

Let us return now to the holomorphic (W, N) with $\Phi: N \to \mathrm{Aut}(T)$ trivial so that to each $\tau \in H^1(N; \mathcal{J})$ for which $\delta(\tau) \in \mathcal{B}$ there is the holomorphic (T, M_τ) with $M_\tau/T = V$. From (3.3) we see that we may alter $\delta(\tau)$ by any element in the image of $0 \to H^2(V; Z^{2k}) \to H^2(N; \mathfrak{Z}^{2k})$. To be certain that this new Bieberbach class is still in the image of $\delta: H^1(N; \mathcal{J}) \to H^2(N; \mathfrak{Z}^{2k})$ we consider the following. Since Φ is trivial, $h_c^0 \to V$ is $\mathcal{O}(V)^k \to V$, the sheaf of germs of holomorphic maps on V into C^k. From the exact sequence $0 \to Z^{2k} \to C \to T \to 0$ we receive a homomorphism $H^2(V; Z^{2k}) \to H^2(V; \mathcal{O}(V)^k)$ and a commutative diagram

$$\begin{array}{ccc} H^2(V; \mathfrak{Z}^{2k}) & & \\ \uparrow & \searrow & \\ H^2(V; Z^{2k}) & \longrightarrow & H^2(V; \mathcal{O}(V)^k) \end{array} .$$

Thus we may alter $\delta(\tau)$ by the image of any element in $H^2(V; Z^{2k})$ which also lies in the kernel of $H^2(V; Z^{2k}) \to H^2(V; \mathcal{O}(V)^k)$.

Since (T, M) has only finite isotropy subgroups there is the Leray spectral sequence of the quotient map $M \to V$ in rational cohomology. Thus there is $\{E_r^{s,t}, d_r\} \Rightarrow H^*(M_\tau; Q)$ with

$$E_2^{s,t} \simeq H^s(V;Q) \otimes H^t(T;Q) \quad .$$

The generators of the exterior algebra $H^*(T;Q)$ are transgressive. Let us digress for a moment to note that by analogy with $\mathfrak{Z}^{2k} \to W$ and $H^*(N;\mathfrak{Z}^{2k})$ we could replace Z by Q to obtain $\mathfrak{Q}^{2k} \to W$ and $H^*(N;\mathfrak{Q}^{2k})$. When the 'E-spectral sequence is considered in this case, however we find $H^*(N;\mathfrak{Q}^{2k}) \simeq H^*(V;Q^{2k})$. Thus the inclusion $\mathfrak{Z}^{2k} \subset \mathfrak{Q}^{2k}$ induces a homomorphisn $H^*(N;\mathfrak{Z}^{2k}) \to H^*(V;Q^{2k})$ the kernel of which is the torsion subgroup.

If we regard $H^2(N;\mathfrak{Z}^{2k})$ as the 2k-fold sum of $H^2(N;\mathfrak{Z})$ with itself then we can write $\delta(\tau) = (a_1, \ldots, a_{2k})$. The image of this element in $H^2(V;Q^{2k})$ is also written $(a_1, \ldots, a_{2k})$. If $v_1, \ldots, v_{2k}$ are the generators of the exterior algebra $H^*(T;Q)$ then $d_2(v_j) = a_j \in H^2(V;Q)$.

Denoting by $b_n(\cdot)$ the n-th betti number of a space we may give as an application a result of Kodaira.

(6.1) Lemma: Suppose $\dim_c W = \dim_c T = 1$ and V is compact. If $\tau \in H^1(N;\mathcal{J})$ is an element for which $\delta(\tau)$ is a Bieberbach class of infinite order then $b_1(M_\tau) = b_3(M_\tau) = b_1(V) + 1$ and $b_2(M_\tau) = 2b_1(V)$.

Proof: Of course d_2 is the only non-trivial differential. Since $(a_1, a_2) \neq 0 \in H^2(V;Q^2)$ it follows that $d_2 : H^1(T;Q) \to H^2(V;Q)$ is an epimorphism and so has a 1-dimensional kernel. One the other hand $H^1(V;Q) \to H^1(M_\tau;Q)$ is a monomorphism. From duality, $b_3(M_\tau) = b_1(M_\tau)$ and since the Euler characteristic of M_τ vanishes we may determine $b_2(M_\tau)$. In this case (5.1) yields

$$0 \to Z + Z \to H^2(N;\mathfrak{Z}^2) \to H^0(V;h^2) \to 0$$

so that there is always a large selection of τ with $\delta(\tau)$ a Bieberbach class of infinite order. We might also note that by Dolbeault's theorem $H^1(V;\Theta(W)^2)$ is the direct sum of $H^{0,1}(V;C)$ with itself. Unless W is closed and simply connected each of the surfaces M_τ is aspherical with fundamental group a torsionless central extension $0 \to Z^2 \to \pi \to N \to 1$.

If V is closed and simply connected then $\delta : H^1(N;\mathcal{J}) \simeq H^2(N;\mathfrak{Z}^2)$ so that every Bieberbach class of infinite order may be uniquely converted in an elliptic surface M_τ in class VII_0 of Kodaira's table [24, p. 790]. The Hopf surface as a Calabi-Eckmann example may be obtained by taking $W = CP(1)$ and N trivial [8]. The reader might like $N = Z_2$. $W = T$. The action is $t \to t^{-1}$ and $V = T/Z_2$ is simply connected. Using (7.5) the reader may prove that each surface in (6.1) fibres smoothly over S^1.

Let us now discuss some analytic analogues of the fibration theorems.

(8.2) Lemma: If $\tau \in H^1(N;\mathcal{J})$ is an element for which $\delta(\tau) \in H^2(N;\mathfrak{Z}^{2k})$ has finite order $n > 0$ then there is a $\tau' \in H^1(N;\mathcal{J})$, also of order n, with $\delta(\tau) = \delta(t')$.

Proof: Since $\delta(n\tau) = 0$ there is a $v \in H^1\left(V;\mathcal{O}(V)^k\right)$ such that $e_*(v) = n\tau$. However, $H^1\left(V;\mathcal{O}(V)^k\right)$ is a vector space so the required element is $\tau' = \tau - e_*(v/n)$.

(8.3) Theorem: Let $n > 0$ be a given integer and let $\tau \in H^1(N;\mathcal{J})$ be an element for which $\delta(\tau) \in \mathcal{B}$. There is a holomorphic map $g\colon M_\tau \to T$ satisfying the identity $g(tx) = t^n g(x)$ for all $t \in T$, $x \in M_\tau$ if and only if $n\tau = 0 \in H^1(N;\mathcal{J})$.

Proof: Suppose first that the g exists. Choose an open covering $\mathcal{U}$ of V and a holomorphic co-ordinate system with operator $\{m_{i,j}(w,\alpha)\}$ representing τ. We form the associated left principal holomorphic T-bundle with operators $(T,B,N) \to (W,N)$. There is the T-equivariant covering map $(T,B) \to (T,M_\tau)$ given by $((t,w,i)) \to [t\ w,i]$. We regard g as being defined on this quotient of B. For each index i we define $\zeta_i(w) = g([1,w,i])$. On $W_i \cap W_j$ this yields

$$g\left[1,w\alpha,j\right] = g\left[m_{i,j}(w\alpha,e),w\alpha,i\right] = g\left[m_{i,j}(w\alpha,e)m_{i,i}(w,\alpha),w,i\right]$$

so that if we apply (4.1) to (i,i,j) with $\alpha = \alpha, \beta = e$ we obtain

$$m_{i,j}(w,\alpha)^n \zeta_i(w) = \zeta_j(w\alpha)$$

proving that $n\tau = 0$.

Conversely, suppose that $n\tau = 0$. We choose a representative holomorphic co-ordinate system with operators $\{m_{i,j}(w,\alpha)\}$ so that there are holomorphic $\zeta_i : W_i \to T$ which satisfy $m_{i,j}(w,\alpha)^n \zeta_i(w) = \zeta_j(w\alpha)$ on $W_i \cap W_j$. We define $g\colon M_\tau \to T$ by $g[t,w,i] = t^n \zeta_i(w)$. Now $[t,w\alpha,j] = [t\,m_{i,j}(w,\alpha),w,i]$, but $t^n \zeta_j(w\alpha)^n = t^n\, m_{i,j}(w,\alpha)^n\, \zeta_i(w)^n$ so that the map g is well defined and has the property $g(tx) \equiv t^n g(x)$.

(8.4) Corollary: If $\tau \in H^1(N;\mathcal{J})$ is an element of finite order for which $\delta(\tau) \in$ then M_τ may be holomorphically fibred over a complex torus T' with a connected fibre and a finite abelian structure group.

Proof: Suppose that τ has order n and choose a holomorphic $g\colon M_\tau \to T$ as in (8.3). We may choose a base point $x_0 \in M_\tau$ with $g(x_0) = e \in T$. The composition $gf^{x_0}\colon T \to T$ is the homomorphism $t \to t^n$. There is also the homomorphism $g_*\colon \pi_1(M_\tau, x_0) \to \pi_1(T)$. Since

$\operatorname{im}(g_*)$ has finite index there is a covering torus $\nu: T' \to T$ to $\operatorname{im}(g_*)$ and a holomorphic $G: M_\tau \to T'$ with $G(x_0) = e \in T'$ and $\nu \circ G = g$. Since $\operatorname{im}(g_*) \supset n\pi_1(T)$ there is also a homomorphism $h: T \to T'$ with $(\nu \circ h)(t) = t^n$. We assert that $G(tx) = h(t)G(x)$. Consider the map $T \times M_\tau \to T'$ given by $(t, x) \to h(t^{-1})G(tx)$. Since ν is a homomorphism $\nu\big(h(t^{-1})G(tx)\big) \equiv e \in T$. Thus $h(t^{-1})G(tx)$ belongs to the finite kernel of ν and so is constant because $T \times M_\tau$ is connected. Taking $t = e$, $x = x_0$ we see that this constant value is $e \in T'$ and so we have established the assertion.

From this assertion it follows that $e \in T'$ is a regular value of G so that $G^{-1}(e) = Y \subset M_\tau$ is a closed submanifold. If $\Delta \subset T$ is the kernel of $h: T \to T'$ then $\Delta Y = Y$ and $tY \cap Y \neq \emptyset$ if and only if $t \in \Delta$. Thus a holomorphic equivalence between $T \times_\Delta Y$ and M_τ is given by $\big((t, y)\big) \to ty$. The fibre Y must be connected since $G_*: \pi_1(M_\tau, e) \to \pi_1(T')$ is an epimorphism.

The following, combining (8.2) and (8.4), is obviously suggested by Kodaira's results on surfaces.

(8.5) Corollary: If $\tau \in H^1(N; \mathcal{J})$ is an element for which $\delta(\tau)$ is a Bieberbach class of finite order then (T, M_τ) can be equivariantly deformed into a $(T, M_{\tau'})$ which fibres as in (8.4).

We should point out that the fibre Y need not even be topologically unique [16]. Carrell has shown that if M_τ is algebraic then τ has finite order. It is further shown in this case that for every deformation τ' of τ, the manifold $M_{\tau'}$ is Kähler.

9. A Smooth 4-dimensional Case

Let us suppose that (W, N) is a properly discontinuous group of diffeomorphisms in which every isotropy subgroup is abelian. In $W \times N$ we consider the subset I of all pairs (w, α) for which $w\alpha = w$. The complement of I is readily seen to be open, so I is closed. There is a right action (I, N) given by $(w, \alpha)\beta = (w\beta, \beta^{-1}\alpha\beta)$. An element in I/N is denoted by $\big((w, \alpha)\big)$. A canonical map $p: I/N \to V$ is given by $p\big((w, \alpha)\big) = \nu(w)$.

(9.1) Lemma: The map $p: I/N \to V$ is a sheaf over V.

Proof: For each $v \in V$ we must define a natural abelian group structure on $p^{-1}(v) \subset I/N$. Select any $w \in W$ for which $\nu(w) = v$. We assert that every element $p^{-1}(v)$ can be written $\big((w, \alpha)\big)$ for a unique $\alpha \in N_w$. Clearly only uniqueness is the issue. Suppose that

$\alpha_j \in N_w$ is an element for which $((w,\alpha)) = ((w,\alpha_1))$, then by definition there is a $\beta \in N$ with $w\beta = w$, $\beta^{-1}\alpha\beta = \alpha_1$. But then $\beta \in N_w$, which is abelian by assumption, so $\alpha = \alpha_1$. The addition in $p^{-1}(v)$ is then defined by $((w,\alpha)) + ((w,\beta)) = ((w,\alpha\beta))$. Suppose $w_1 \in W$ is an element for which $\nu(w_1) = v$ also, then there is a $\gamma \in N$ with $w\gamma = w_1$. But then $((w,\alpha)) = ((w_1,\gamma^{-1}\alpha\gamma))$, $((w,\beta)) = ((w_1,\gamma^{-1}\beta\gamma))$ so that $((w_1,\gamma^{-1}\alpha\gamma)) + ((w_1,\gamma^{-1}\beta\gamma)) = ((w_1,\gamma^{-1}(\alpha\beta)\gamma)) = ((w,\alpha\beta))$. Thus we have shown that the abelian group structure on $p^{-1}(v)$ is well defined. Obviously at each $w \in W$ there is a canonical isomorphism of N_w with $p^{-1}(\nu(w))$. We therefore refer to $p: I/N \to V$ as <u>the isotropy sheaf</u>.

Let us now impose more stringent requirements on (W,N). As before, we denote by $S \subset W$ the subset of points at which the abelian isotropy subgroup is non-trivial. From this point onward we suppose

(*) <u>the quotient</u> $S/N \subset V$ <u>is finite</u>.

Thus S is a discrete closed subset of W and it follows immediately that every non-trivial isotropy subgroup of (W,N) is cyclic.

The isotropy sheaf $p: I/N \to V$ is trivial on the complement of the finite set S/N. Let us use this to define a homomorphism $H^0(V; I/N) \to N/[N,N]$ on the sections of the isotropy sheaf into the commutator quotient group. If $\mu: N \to N/[N,N]$ denotes the quotient homomorphism then a well defined function $\mu: I/N \to N/[N,N]$ is given by $\mu((w,\alpha)) = \mu(\alpha)$. Then if $\chi: V \to I/N$ is a section $\mu(\chi) \in N/[N,N]$ is defined by $\sum_{v \in V} \mu(\chi(v))$. The isotropy sheaf is non-trivial only over the finite set S/N, thus this yields a well defined homomorphism $H^0(V, I/N) \to N/[N,N]$.

Let us take $\Phi: N \to GL(k, Z)$ to be trivial and consider $H^*(N; \mathfrak{Z})$. In the 'E-spectral sequence each $h^j \to V$ is trivial if j is odd and trivial on the complement of the finite set S/N if j is even and positive. Accordingly we can state (compare [13, (11.2)])

(9.2) <u>Lemma</u>: <u>Let</u> (W,N) <u>be a properly discontinuous group of diffeomorphisms for which</u> $S/N \subset V$ <u>is finite</u>. <u>For every</u> $j > 0$ <u>there is an exact</u>

sequence

$$0 \to H^{2j}(V; Z) \to H^{2j}(N; \mathfrak{Z}) \to H^{0}(V; h^{2j}) \xrightarrow{d_{2j+1}} H^{2j+1}(V; Z) \to H^{2j+1}(N; \mathfrak{Z}) \to 0 .$$

Proof: In the 'E-spectral sequence $E_2^{i,j} = 0$ if j is odd or if $i > 0$ and j is even. Now each isotropy subgroup is cyclic so from (2.2) we see that for $j > 0$, $h^{2j} \to V$ is isomorphic to the isotropy sheaf $I/N \to V$. Let us see if we can exploit this informal observation.

We shall now restrict our attention to a group of orientation preserving diffeomorphisms on an oriented manifold for which S/N is finite and V is compact. The preservation of orientation implies $\dim W = 2n$. The quotient V is a compact manifold with finitely many singular points. Each singular point has a neighborhood which is the cone over a $2n-1$ dimensional lens space. We see this as follows. If $v_1, \ldots, v_k$ are the points in S/N we select $w_1, \ldots, w_k$ in W with $\nu(w_j) = v_j$. At each w_j we center a closed invariant $2n$-cell K_j such that

(i) $(K_j \cdot N) \cap (K_i \cdot N) = \emptyset$ if $i \neq j$

(ii) $K_j \cap K_j \alpha \neq \emptyset$ if and only if $\alpha \in N_{w_j}$

(iii) N_{w_j} acts orthogonally on K_j.

Then $K_j/N_{w_j} \subset V$ is the cone over the lens space $\partial K_j/N_{w_j}$. Let us put $D_j = K_j \cdot N = K_j \times_{N_{w_j}} N$ and $D = \bigcup_{j=1}^{K} D_j$. Then we may write $W = B \cup D$ where $B = W \setminus D^0$, $B \cap D = \partial B = \partial D = \bigcup \partial D_j$. Now B is a closed (as a subset) N-invariant submanifold for which the induced action (B, N) has only trivial isotropy groups. Furthermore let us note that $H^i(D; Z) \simeq H^i(\partial D; Z)$, $0 \leq i \leq 2n-2$. We introduce the Borel spaces

$$\begin{tikzcd} D \times_N X \arrow[dr] & & \partial D \times_N X \arrow[ll] \arrow[dl] \\ & X/N & \end{tikzcd}$$

and it follows immediately that

$$H^i(D \times_N X; Z) \simeq H^i(\partial D \times_N X; Z), \qquad 0 \leq i \leq 2n-2$$

$$H^{2n-1}(D \times_N X; Z) \subset H^{2n-1}(\partial D \times_N X; Z) .$$

If we write $W \times_N X = (B \times_N X) \cup (D \times_N X)$ and put the preceding information into the resulting Mayer-Vietoris sequence we learn that $H^i(N; \mathfrak{Z}) \simeq H^i(B \times_N X; Z)$ for $0 \leq i \leq 2n-2$. However (B, N) is a group of covering transformations so that $B \times_N X \to B/N$ is also a fibration with contractible fibre X.

(9.3) Lemma: Let (W, N) be a properly discontinuous group of orientation preserving diffeomorphisms on an oriented even dimensional manifold with $2n > 2$. If V is compact and S/N is finite then for $0 \leq i \leq 2n-2$

$$H^i(B/N; Z) \simeq H^j(N; \mathfrak{Z}) .$$

Now B/N is just a compact oriented manifold with boundary the finite disjoint union of lens spaces $\partial K_j/N_{w_j}$. Put $B^* = B/N$ and consider

$$\cdots \to H^{2n-2}(B^*; Z) \xrightarrow{i^*} H^{2n-2}(\partial B^*; Z) \to \cdots$$

But is this not the edge homomorphism

$$H^{2n-2}(N; \mathfrak{Z}) \to H^0(V; h^{2n-2})$$

since surely $H^{2n-2}(\partial B^*; Z) \simeq H^0(V; h^{2n-2})$? There is an orientation class $\sigma \in H_{2n-1}(\partial B^*; Z)$ and for any $a \in H^{2n-2}(B^*; Z)$ we have

$$i_*\left(i^*(a)\cap\sigma\right) = 0 \in H_1(B; Z) .$$

Thus the image of $H^{2n-2}(B^*; Z) \to H^{2n-2}(\partial B^*; Z)$ is dual to the kernel of $H_1(\partial B^*; Z) \to H_1(B^*; Z)$.

Suppose that W simply connected, then since $2n > 2$ we see B is also simply connected so that $\pi_1(B^*) \simeq N$ and $H_1(B^*; Z) \simeq N/[N,N]$. At each w_j we identify the cyclic group N_{w_j} with $\pi_1(\partial K_j/N_{w_j}) \simeq H_1(\partial K_j/N_{w_j}; Z)$. In this way $H^0(V; I/N)$ is identified with $H_1(\partial B^*; Z)$ and the canonical $\chi: H^0(V; I/N) \to N/[N,N]$ with $H_1(\partial B^*; Z) \to H_1(B^*; Z)$. The identification of N_{w_j} with $\pi_1(\partial K_j/N_{w_j})$ depends on choosing a generator.

(9.4) Theorem: Let (W, N) be a properly discontinuous group of orientation preserving diffeomorphisms on a simply connected oriented manifold. If $\dim W = 2n > 2$ and if $S/T \subset V$ is finite then there is an isomorphism $H^0(V; h^{2n-2}) \simeq H^0(V; I/N)$ which identifies the image of the edge homomorphism

$$H^{2n-2}(N; \mathfrak{Z}) \to H^0(V; h^{2n-2})$$

with the kernel of

$$\mu: H^0(V; I/N) \to N/[N,N] .$$

In practice the identification is obvious. The result achieves its maximum significance when $\dim W = 4$.

(9.5) Corollary: If (W, N) is a properly discontinuous group of orientation preserving diffeomorphisms on a simply connected 4-dimensional manifold for which V is compact and S/N is finite, then the image of $H^2(N; \mathfrak{Z}) \to H^0(V; h^2)$ is identified with the kernel of $\mu: H^0(V; I/N) \to N/[N,N]$. Furthermore $\mathcal{B} \subset H^2(N; \mathfrak{Z})$ is non-empty if and only if there is a selection of generators $\alpha_j \in N_{w_j}$ for which $\alpha_1\alpha_2 \ldots \alpha_k \in [N,N]$.

An element $a \in H^2(N; \mathfrak{Z})$ lies in $\mathcal{B}$ if and only if its image in each stalk $h^2_{v_j} \simeq H^2(N_{w_j}; Z) \simeq N_{w_j}$ is a generator of the cyclic group.

If $W/N = V$ is also simply connected then N is the least normal subgroup containing all the isotropy groups and hence in this case $\mu: H^0(V; I/N) \to N/[N, N]$ is an epimorphism.

Let us consider (Z_p, M) a cyclic group of prime order acting as a group of orientation preserving diffeomorphisms with a finite non-empty fixed point set on a closed oriented aspherical 4-manifold. Select a fixed point $x \in M$ and denote by τ_* the induced automorphism of $\pi_1(M, x)$ for each $\tau \in Z_p$. The semi-direct product $N = \pi_1(M, x) \circ Z_p$ then acts on the contractible universal covering space W. Thus (W, N) satisfies the hypothesis of (9.5).

We define the set $H^1(Z_p; \pi)$ to be the set of crossed-homomorphisms $\phi: Z_p \to \pi$ identified with respect to the equivalence $\phi \sim \phi_1$ if and only if there is an $\alpha \in \pi$ for which

$$\phi_1(\tau) = \alpha\phi(\tau)\tau_*(\alpha^{-1})$$

for all $\tau \in Z_p$. We have shown [17, A.10], that the points in the fixed point set of (Z_p, M) are in 1-1 correspondence with the elements of the cohomology set $H^1(Z_p, \pi)$.

Now $N/[N, N] = H_1(N; Z)$, and since $N = \pi \circ Z_p$ we may conclude that

$$N/[N, N] = Z_p \oplus H_0\left(Z_p; H_1(\pi; Z)\right).$$

Let k be the number of fixed points. Choose representatives $\phi_1, \ldots, \phi_k : Z_k \to \pi$ for each of the elements in $H^1(Z_p; \pi)$. Then $\mu: (Z_p)^k \to Z_p \oplus H_0\left(Z_p; H_1(\pi; Z)\right)$ is given by

$$(\tau_1, \ldots, \tau_k) \to \left(\sum \tau_j, \sum \phi_j(\tau_i)\right)$$

This second sum is taken in the group $H_0\left(Z_p; H_1(\pi; Z)\right)$ by composing

$$\pi \to H_1(\pi; Z) \to H_0\left(Z_p; H_1(\pi; Z)\right),$$

thus μ is independent of the representatives ϕ_j which are chosen, since $\alpha\phi(\tau)\tau_*(\alpha^{-1}) = \phi(\tau) \in H_0\left(Z_p; H_1(\pi; Z)\right)$. The kernel of this μ is the image of

$$H^2(N; Z) \to H^0(V; h^2).$$

If T is a complex toral group with $\dim_c T = 2$ then Z_2 acts by $t \to t^{-1}$ and $T/Z_2 = V$ is simply connected. There are 16 fixed points; $H_0\left(Z_2; H_1(\pi; Z_2)\right) \simeq (Z_2)^4$ and $\mu: (Z_2)^{16} \to Z_2 \oplus (Z_2)^4$ is an epimorphism. The ϕ_j may be described as follows. There are 16 ordered 4-tuples of integers containing only 0 and 1. Lexicographically order these. Let

$\phi_j : Z_2 \to Z^4$ be the crossed-homomorphism which to the generator of Z_2 assigns in Z^4 the j^{th} term in this lexicographic ordering. Thus ϕ_1 is trivial.

This calculation is independent of the complex structure on T. We know $H^*(V; C) \simeq H^*(T/Z_2; C)$ is isomorphic to the subalgebra of elements of $H^*(T; C)$ that are fixed under the induced representation of Z_2. Thus $H^2(V; C) \simeq H^2(T; C)$ and in fact $H^2(V; \mathcal{O}(V)) \simeq H^2(T; \mathcal{O}(T)) \simeq H^{0,2}(T; C)$. Taking $k = 1$, $N = \pi_1(T) \circ Z_2$ and Φ trivial we have

$$0 \to H^1(N; \mathcal{Y}) \to H^2(N; Z+Z) \to H^{0,2}(T; C) .$$

A Bieberbach class in $\mathcal{B} \subset H^2(N; Z+Z)$ is uniquely determined modulo the sum with an element in the image of $0 \to H^2(V; Z+Z) \to H^2(N; Z+Z)$ as follows. Divide the above lexicographically ordered set into the subsets S_1, S_2 (which may overlap and one may be $\emptyset$) subject to

(i) the cardinality of each of the two subsets is even

(ii) in each of the two subsets the sum of the elements is $0 \bmod 2$.

We obtain two elements in the kernel of $\mu : (Z_2)^{16} \to Z_2 \oplus (Z_2^4)$; $a_1 = (\tau_1, \ldots, \tau_{16})$ with τ_j the generator of Z_2 if and only if $j \in S_1$, $a_2 = (\tau_1', \ldots, \tau_{16}')$ with τ_j' the generator of Z_2 if and only if $j \in S_2$. Corresponding to the ordered pair (a_1, a_2) there is a Bieberbach class in $H^2(N; Z+Z)$ with (a_1, a_2) its image in $H^0(V; h^2)$.

From $Z+Z \to C$ we have

$$\begin{array}{ccccc} H^2(T; Z+Z) & \longrightarrow & H^2(T; C) & \longrightarrow & H^{0,2}(T; C) \\ \uparrow & & \uparrow \simeq & & \uparrow \simeq \\ H^2(V; Z+Z) & \longrightarrow & H^2(V; C) & \longrightarrow & H^{0,2}(V; C) \end{array}$$

which we use to find elements in $H^2(V; Z+Z)$ by whose image in $H^2(N; Z+Z)$ a Bieberbach class with holomorphic realization can be translated into another Bieberbach class which still has a holomorphic realization. The <u>resulting closed complex</u> 3-folds <u>will all be aspherical</u>. We thought the reader might like some idea of this construction technique as applied to a specific example. Incidentally, there is at least one Bieberbach class of finite order present in $H^2(N; Z+Z)$ for this example, but we have no technique for counting Bieberbach classes of finite order when $\dim_c W > 1$.

9.6 We shall now give another class of locally injective examples where the resulting manifolds will have finite fundamental groups. We recall that after Theorem 7.4 we treated two types of examples. One was the manifolds of constant positive curvature. Let us now look at a finite group F which has a fixed point free unitary representation of minimal degree 3. The group that we shall investigate is an extension

$$1 \longrightarrow Z_7 \longrightarrow F \longrightarrow Z_9 \longrightarrow 1 .$$

$$F = \left\{a, b \,\middle|\, a^7 = b^9 = 1, \ \ bab^{-1} = a^2\right\} .$$

This group can be represented by the matrices

$$a \longrightarrow A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^4 \end{pmatrix} . \qquad b \longrightarrow B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \beta & 0 & 0 \end{pmatrix} .$$

where $\alpha = \exp(2\pi i k/7)$, $(k, 7) = 1$; and $\beta = \exp(2\pi i k'/3)$, $(k', 3) = 1$.

The center K clearly is the cyclic group $\approx Z_3$ generated by b^3. We denote the quotient F/K by N. We may present it as:

$$N = \left\{'a, 'b \,\middle|\, 'a^7 = 1, \ \ 'b^3 = 1, \ \ 'b'a'b^{-1} = 'a^2\right\}$$

This is a non-abelian split extension

$$1 \longrightarrow Z_7 \longrightarrow Z_7 \circ Z_3 \rightleftarrows Z_3 \longrightarrow 1 .$$

This group N is a metacyclic group and has periodic cohomology of period 6. The cohomology can be computed easily and is $H^i(N; Z) = Z, 0, Z_3, 0$ in dimensions $0, 1, 2,$ and 3, respectively. (For example, $N/[N, N] = Z_3$, with $[N, N]$ being Z_7. This then implies that $H^2(N; Z) \cong Z_3$ since it is $\mathrm{Ext}\big(H_1(N; Z), Z\big)$. Using duality $H^2(N; Z) \approx H_3(N; Z) \approx Z_3$ and $H^3(N; Z) = \mathrm{Ext}\big(H_4(N; Z), Z\big) \cong \mathrm{Ext}\big(H^1(N; Z), Z\big) = 0$.)

As in § 7 we use the $'E$ and $''E$ spectral sequences of low degree. We have

$$\begin{tikzcd}
 & & {''E_2^{0,1}} \arrow[d, "\delta"] & & \\
 & & {''E_2^{2,0}} \arrow[d] & & \\
{'E_2^{0,1}} \arrow[r, "\delta"] & {'E_2^{2,0}} \arrow[r, "i"] & H^2(N;\mathfrak{z}) \arrow[r, "j"] \arrow[d, "h"] & {'E_2^{0,2}} \arrow[r] & {'E_2^{0,3}} \arrow[dd] \\
 & H^0\left(N; H^2(\mathbb{C}P_2; Z)\right) = & {''E_2^{0,2}} \arrow[d] & & \\
 & & {''E_2^{3,0}} \arrow[rr] & & H^3(N;\mathfrak{z}) \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

which becomes

$$\begin{tikzcd}
 & & 0 \arrow[d, "\delta"] & & \\
 & & Z_3 \arrow[d] & & \\
0 \arrow[r, "d"] & H^2(CP_2/N; Z) \arrow[r, "i"] & H^2(N; \mathfrak{z}) \arrow[r, "j"] \arrow[d, "h"] & Z_7 \oplus Z_3 \oplus Z_3 \oplus Z_3 \arrow[r] & {'E_2^{0,3}} \arrow[dd] \\
 & & Z \arrow[d] & & \\
 & & 0 & & H^3(N; \mathfrak{z}) \arrow[d] \\
 & & & & 0 .
\end{tikzcd}$$

We have yet to explain why $'E_2^{0,2} \cong Z_7 \oplus Z_3 \oplus Z_3 \oplus Z_3$, and the $'E_2^{2,0}$ and $''E_2^{0,2}$ terms. The group N acts trivially on the cohomology of $H^2(CP_2; Z)$ and thus $''E_2^{0,2}$ is isomorphic to Z. We claim now that $H^2(CP_2/N; Z) \cong Z$. We see this by observing that CP_2/N is simply connected, consequently the torsion subgroup of $H^2(\mathbb{C}P_2/N; Z)$ is trivial. To see that CP_2/N is simply connected we observe that (CP_2, N) is the composition of the actions $(CP_2, Z_7) \xrightarrow{/Z_7} (CP_2/Z_7, N/Z_7) \xrightarrow{/Z_3} CP_2/N$. Z_7 has fixed points and by the Lefschetz formula so does $(CP_2/Z_7, Z_3)$. We also may see this by direct computation with matrices A and B. The group $H^2(N;\mathfrak{z})$ is clearly $Z \oplus Z_3$. Since $H_1(N; Z) = Z_3$, the cokernel of $H^2(N;\mathfrak{z}) \longrightarrow {'E_2^{0,2}}$ is Z_3 by 9.5, provided that we have checked that all the isotropy subgroups of (CP_2, N) are isolated. This can be done by looking at the matrices A and B with homogeneous coordinates. One finds directly that there is exactly one orbit

of points in CP_2 whose stability group is Z_7 and there are three distinct orbits on which the stability group is Z_3. (This is a special case of a much more general result of Dost Khan [22], in which he explicitly computes all the stability groups on CP_n for the induced actions of free linear representations.)

Our diagram now becomes the exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & Z_3 & & & & & & \\ & & \downarrow & & & & & & \\ 0 \longrightarrow Z & \xrightarrow{i} & Z \oplus Z_3 & \xrightarrow{j} & Z_7 \oplus Z_3 \oplus Z_3 \oplus Z_3 & \xrightarrow{\mu} & Z_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & & & & & \\ & & Z & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

It is clear that j restricted to Z must span $Z_7 \oplus Z_3$. Thus the image of i must be 21 times a free generator. The commutator subgroup is Z_7. Now we may explicitly calculate that there are generators $x_j \in N_{w_j}$ for which $x_1x_2x_3x_4 \in [N, N]$; or we may use the general fact that we do get a Bieberbach class by choosing the locally injective coordinate bundle with linear operators:

$$\left(U(1)/K,\ S^5/K,\ F/K\right) \xrightarrow{/U(1)/K} (CP_2, N)\ .$$

For this case, the characteristic class of the bundle over CP_2 is three times the generator in $H^2(CP_2; Z) = H^0\left(N; H^2(CP_2; Z)\right)$.

We would now like to determine some of the Bieberbach classes in $H^2(N;\mathfrak{Z}) \cong Z \oplus Z_3$. The torsion of $H^2(N;\mathfrak{Z})$ can never be a Bieberbach class since its image under j avoids non-trivial elements of Z_7. Let us write $(1, 0)$ and $(0, \omega)$ for generators of $Z \oplus Z_3$. We choose ω so that $(3, \omega)$ yields the Bieberbach class represented by the "linear" $\left(U(1)/K, S^5/F\right)$ which has been discussed above. We write the image of $(1, 0)$ under j by $(\gamma, \delta_1, \delta_2, \delta_3)$ and of $(0, \omega)$ by $(0, \omega_1, \omega_2, \omega_3)$. Then the image of $(3, \omega) =$ $(3\gamma, \omega_1 + 3\delta_1, \omega_2 + 3\delta_2, \omega_3 + 3\delta_3) = (\gamma, \omega_1, \omega_2, \omega_3)$, a Bieberbach class. Thus $\omega_1, \omega_2, \omega_3$ are each generators of the distinct conjugacy classes of the $N_{w_1}, N_{w_2}, N_{w_3}$ of order 3. Furthermore, the image of $(1, 0)$ cannot be $(\gamma, 0, 0, 0)$. At least, δ_1, say, is not 0. One strongly suspects that $(\pm 1, 0)$ or $(\pm 1, \pm\omega)$ could never be a Bieberbach class. (If they

were this would yield a free action of N on the 5-sphere which commutes with the linear free diagonal $U(1) \subset U(3)$ and which projects to a linear action on CP_2. It is known that the free (S^5, N) could not be equivalent to a linear action. However, such non-linear smooth free actions do actually exist according to the recent results of R. Lee and T. Petrie. It would be surprising if they were to arise this way.) In any case it is clear now that at least $(3k, \pm\omega)$, where $k \not\equiv 0(7)$ are all Bieberbach classes in $H^2(N; \mathfrak{Z})$. Each of these classes yields a 5-manifold which should be a manifold of constant positive curvature and of minimal dimension.

We may also form holomorphic examples by using C^* principal bundles instead of $U(1)$-bundles. Then dividing out by $Z \subset C^*$ we obtain a complex torus and Bieberbach classes in $H^2(N; \mathfrak{Z}^2)$, diffeomorphic to $S^1 \times M^5$, where M^5 is a smooth 5-manifold just constructed. The cohomology groups $H^1(CP_2/N; h_c^0)$ are all trivial, for $j > 0$. There are, in particular, no obstructions to holomorphic realizations of Bieberbach classes, nor are there any deformations possible.

10. Holomorphically Injective Seifert Fiberings

We denote by $h^{1,0}(M)$ the space of closed holomorphic 1-forms on a closed connected manifold and by $\mathrm{Re}: h^{1,0}(M) \to H^1(M;R)$ the real linear embedding which to each element of $h^{1,0}(M)$ assigns the real parts of its periods. Suppose now (T, M) is an effective holomorphic action. At each $x \in M$ the map $f_x: T \to M$ given by $f_x(t) = tx$ produces a commutative diagram

$$\begin{tikzcd} h^{1,0}(T) \arrow[d, "\simeq"', "\mathrm{Re}"] & h^{1,0}(M) \arrow[l, "f_x^*"'] \arrow[d, "\mathrm{Re}"] \\ H^1(T;R) & H^1(M;R) \arrow[l, "f^*"'] \end{tikzcd}$$

If $f^*: H^1(M;R) \to H^1(T;R)$ is an epimorphism then the action (T, M) is homologically injective and M topologically fibres over a quotient of T by a finite subgroup, which is also the structure group of the fibration. Furthermore the fibre is connected. By analogy, let us say (T, M) is holomorphically injective if and only if $f_x^*: h^{1,0}(M) \to h^{1,0}(T)$ is an epimorphism. While this will not imply that M holomorphically fibres over a quotient of T, it does have important consequences. We wish to study this holomorphically injective condition in this section. We shall work with holomorphic Seifert fiberings.

Let M denote a closed connected analytic manifold and let $\Psi: \pi_1(M) \to GL(k, C)$ be a homomorphism. We wish to define $h^{1,0}(M, \Psi)$. We denote by (M^*, π) the universal covering of M. For each $\alpha \in \pi$ we set $\Psi(\alpha) = \alpha_* \in GL(k, C)$. Via the homomorphism, C^k becomes a $Z(\pi)$-module. The usual $Z(\pi)$-module structure is given map(M^*, C^k); that is, $\alpha_\#(f)(x) = \alpha_*\big(f(x\alpha)\big)$. Now we select a base point $x_0 \in M^*$ and denote by $\mathrm{map}_0(M^*, C^k)$ the linear space of all holomorphic $g: M^* \to C^k$ for which $g(x_0) = 0$. This too may be given a $Z(\pi)$-module structure by $\big(\alpha_\#(g)\big)(x) = \alpha_*\big(g(x\alpha)\big) - \alpha_*\big(g(x_0\alpha)\big)$. We must verify the composition rule.
$\Big(\alpha_\#\big(\beta_\#(g)\big)\Big)(x) = \alpha_*\big(\beta_\#(g)(x\alpha)\big) - \alpha_*\big(\beta_\#(g)(x_0\alpha)\big) = \alpha_*\big(\beta_*(gx\alpha\beta)\big) - \alpha_*\big(\beta_*(gx_0\beta)\big) - \alpha_*\big(\beta_* g(x_0\alpha\beta)\big)$
$+ \alpha_*\big(\beta_* g(x_0\beta)\big) = \big((\alpha\beta)_\#(g)\big)(x)$.

While $\mathrm{map}_0(M^*, C^k)$ is not a submodule of $\mathrm{map}(M^*, C^k)$, there is a short exact sequence of $Z(\pi)$-modules

$$0 \to C^k \to \mathrm{map}(M^*, C^k) \to \mathrm{map}_0(M^*, C^k) \to 0$$

where the second homomorphism is given by $g(x) \equiv f(x) - f(x_0)$. Again we must compute

$$\alpha_{\#}(g)(x) = \alpha_*\big(g(x\alpha)\big) - \alpha_*\big(g(x_0\alpha)\big)$$

$$= \alpha_*\big(f(x\alpha)\big) - \alpha_*\big(f(x_0)\big) - \alpha_*\big(f(x_0\alpha)\big) + \alpha_*\big(f(x_0)\big)$$

$$= \alpha_{\#}(f)(x) - \alpha_{\#}(f)(x_0)$$

to see that this is a $Z(\pi)$-module homomorphism.

There is the associated exact cohomology triangle

$$\begin{tikzcd} H^*_\Psi(\pi; C^k) \arrow[rr] & & H^*_\Psi\big(\pi; \mathrm{map}(M^*, C^k)\big) \arrow[dl] \\ & H^*_\Psi\big(\pi; \mathrm{map}_0(M^*, C^k)\big) \arrow[ul] & \end{tikzcd}$$

The invariant with which we shall be concerned, $h^{1,0}(M,\Psi)$, is the kernel of $H^1_\Psi(\pi; C^k) \longrightarrow H^1_\Psi\big(\pi; \mathrm{map}(M^*, C^k)\big)$. Since π is finitely generated, $H^1_\Psi(\pi; C^k)$ is finite dimensional and hence so is $h^{1,0}(M,\Psi)$. A cocycle $\phi \in Z^1_\Psi(\pi; C^k)$ is a crossed-homomorphism. The cocycle represents an element of $h^{1,0}(M,\Psi)$ if and only if there is a holomorphic $f: M^* \to C^k$ for which $\alpha_*\big(f(x\alpha)\big) - f(x) \equiv \phi(\alpha)$; that is, $\phi \in Z^1_\Psi\big(\pi; \mathrm{map}(M^*, C^k)\big)$ becomes a principal crossed-homomorphism. Since the image of $\delta: H^0_\Psi\big(\pi; \mathrm{map}_0(M^*, C^k)\big) \to H^1_\Psi(\pi; C^k)$ is $h^{1,0}(M,\Psi)$ we can for each element of $h^{1,0}(M,\Psi)$ select a representative crossed-homomorphism, ϕ, together with a holomorphic $g: M^* \to C^k$ with $g(x_0) = 0$ and $\alpha_*\big(g(x\alpha)\big) - g(x) \equiv \phi(\alpha)$.

The functorial property of $h^{1,0}(M,\Psi)$ may be described as follows. If $F: M \to M_1$ is a holomorphic map for which the diagram

$$\begin{tikzcd} \pi_1(M,x) \arrow[rr, "F_*"] \arrow[dr, "\Psi"'] & & \pi_1(M_1,x_1) \arrow[dl, "\Psi_1"] \\ & GL(k,C) & \end{tikzcd}$$

commutes, then there is induced a homomorphism $F^*: h^{1,0}(M_1,\Psi_1) \to h^{1,0}(M,\Psi)$. This is defined by introducing the map $F: M^* \to M_1^*$, covering F, and for which $F(x\alpha) \equiv f(x)F_*(\alpha)$.

If $k = 1$ and Ψ is trivial then $h^{1,0}(M, tr) = h^{1,0}(M)$, the space of closed holomorphic 1-forms. If $\Psi: \pi \to GL(k, C)$ is trivial, then of course $h^{1,0}(M, tr)$ is the k-fold sum of $h^{1,0}(M)$ with itself. We note for later reference

(10.1) <u>Lemma</u>: <u>If</u> T <u>is a complex toral group and if</u> $\Psi: \pi_1(T) \to GL(k, C)$ <u>is trivial then</u> $h^{1,0}(T, tr)$ <u>is isomorphic to</u> $\mathrm{Hom}_c(C^k, C^k)$.

Now let us turn to Seifert fiberings. Let (W, N) be a properly discontinuous group of holomorphic transformations on a simply connected analytic manifold for which $V = W/N$ is compact. We choose a torus

$$0 \to Z^{2k} \xrightarrow{\epsilon} C^k \xrightarrow{e} T \to 0$$

together with a homomorphism $\Phi: N \to \mathrm{Aut}(T)$. Then to any $\tau \in H^1(N; \mathfrak{J})$ for which $\delta(\tau) \in H^2(N; \mathfrak{Z}^{2k})$ is a Bieberbach class there is associated a holomorphic left principal T-bundle with right operators, $(T, B, N) \to (W, N)$, such that

a) (B, N) is a properly discontinuous group of holomorphic covering transformations

b) $(tb)\alpha = \alpha_*^{-1}(t)(b\alpha)$, all $t \in T$, $b \in B$ and $\alpha \in N$.

The quotient $B/N = M_\tau$ is a closed non-singular manifold together with a canonical holomorphic Seifert fibering over V.

At any point $b \in B$ there is $f_b: T \to B$ given by $f_b(t) = tb$. Since W is simply connected, f_b induces an epimorphism $\pi_1(T) \to \pi_1(B, b)$. We identify Z^{2k} with $\pi_1(T)$ via $0 \to Z^{2k} \to C^k \to T \to 0$, then Z^{2k} is given a $Z(N)$-module structure by $\epsilon(\alpha_*(p)) = \alpha_*(\epsilon(p))$ for all $p \in Z^{2k}$. The kernel of $Z^{2k} \to \pi_1(B, b)$ is a submodule, for if $p \in Z^{2k}$ is represented by a closed loop $\sigma(\tau)$ in T, then $\alpha_*(p)$ is represented by $\alpha_*(\sigma(\tau))$, and $\alpha_*(\sigma(\tau)) \cdot b = (\sigma(\tau)b\alpha)\alpha^{-1}$ so that if $f_*^b(p) = 0$, then $f_*^b(\alpha_*(p)) = 0$ also. Thus there is induced a $Z(N)$-module structure on $\mathrm{im}(f_*^b) = \pi_1(B, b)$. <u>This abstract kernel is realized by the group extension</u>

$$0 \to \pi_1(B) \to \pi_1(M_\tau) \to N \to 1 \quad .$$

Let $\psi: \pi \to \mathrm{Aut}(T) \subset GL(k, C)$ be the composition $\pi_1(M) \to N \to \mathrm{Aut}(T)$, then for every $p \in Z^{2k}$, $\alpha \in \pi$

$$f_*^b(\alpha_*(p)) = \alpha f_*^b(p)\alpha^{-1} \quad .$$

Of course $\mathrm{im}(f_*^b) \subset \ker(\psi)$. We may now consider the induced

$$H^1_\psi(\pi; C^k) \to \mathrm{Hom}(Z^{2k}, C^k) \quad .$$

If $\phi: \pi \to C^k$ is a crossed-homomorphism then $\hat{\phi}(p) = \phi(f^b(p))$ is the induced homomorphism. In particular we observe that

$$\alpha_* \hat{\phi}\left(\alpha_*^{-1}(p)\right) = \alpha_* \left(\phi\left(\alpha^{-1} f_*^b(p)\alpha\right)\right)$$

$$= \alpha_* \phi(\alpha^{-1}) + \phi\left(f_*^b(p)\right) + \phi(\alpha)$$

$$= \hat{\phi}(p) \ .$$

(10.2) Lemma: The image of $H^1_\Psi(\pi; C^k) \to \mathrm{Hom}(Z^{2k}, C^k)$ lies in the subspace $\mathrm{Hom}(Z^{2k}, C^k)^\pi$, the subspace of all homomorphisms which satisfy $\alpha_* \hat{\phi}\left(\alpha_*^{-1}(p)\right) \equiv \hat{\phi}(p)$.

This is a special case of [25, Ch. XI; Lemma 9.1].

Now if $F: T \to M_\tau$ is the composition of $T \xrightarrow{f_b} B \to M_\tau$, then F maps T holomorphically onto a fibre. We shall be concerned with the induced

$$F^*: h^{1,0}(M, \Psi) \to \mathrm{Hom}_c(C^k, C^k) \ .$$

In view of (10.2) the image lies in the subspace of all linear transformations which satisfy $\alpha_* \circ L \circ \alpha_*^{-1} = L$, all $\alpha \in \pi$. Simply note that if $\hat{\phi}(p) = L\left(\epsilon(p)\right)$ is in the image of $H^1_\Psi(\pi; C^k) \to \mathrm{Hom}(Z^{2k}, C^k)$ then $\alpha_* L\left(\alpha_*^{-1} \epsilon(p)\right) \equiv L\left(\epsilon(p)\right)$. The image of ϵ spans C^k, so it follows L commutes with every α_*.

(10.3) Definition: The Seifert fibration $M_\tau \to V$ is holomorphically injective if and only if $h^{1,0}(M_\tau, \Psi) \to \mathrm{Hom}_c(C^k, C^k)^\pi$ is an epimorphism.

Let us note that the identity matrix will then belong to the image of the induced homomorphism. We shall need a construction before we can proceed. Let (M^*, π) be the universal cover of M_τ. Then M^* is also the universal cover, as defined in section 3, of (B, N). We assert there is a holomorphic action with operators (C^k, M^*, π) which covers (T, B, N) and for which

a) $(v \cdot x)\alpha = \alpha_*^{-1}(v) \cdot x\alpha, \ v \in C^k, \ x \in M^*, \ \alpha \in \pi$

b) $\epsilon(p) \cdot x = x \cdot f_*^b(p)$, all $p \in Z^{2k}$.

By $e: C^k \to T$ we see that C^k acts on B. Since C^k is simply connected we may apply [13, Th. 4.3] to obtain the covering action (C^k, M^*). The two stated properties relating (C^k, M^*) to (M^*, π) now follow easily from the lemma [13, Lemma 4.1].

(10.4) Lemma: If $M_\tau \to V$ is holomorphically injective then the left principal $(T, B) \to W$ admits a global holomorphic cross-section.

Proof: Choose a base point $x_0 \in M^*$ above b_0. There is a crossed-homomorphism $\phi: \pi \to C^k$ and a holomorphic $G: M^* \to C^k$ such that

$$\text{(i)}\quad G(x_0) = 0, \quad \alpha_* G(x\alpha) - G(x) \equiv \phi(\alpha)$$

$$\text{(ii)}\quad \phi\left(f_*^{b_0}(p)\right) \equiv \epsilon(p)$$

Now for each fixed $x \in M^*$ consider the holomorphic $L_x(v) = G(vx) - G(x)$. Obviously $L_x(0) = 0$. Moreover

$$L_x\left(v + \epsilon(p)\right) = G\left(\epsilon(p)(v \cdot x)\right) - G(x) = G\left((v \cdot x) f_*^{b_0}(p)\right) - G(x).$$

Since $f_*^{b_0}(p)$ is in the kernel of ψ, however,

$$G\left((v \cdot x) f_*^{b_0}(p)\right) - G(v \cdot x) = \phi\left(f_*^{b_0}(p)\right) = \epsilon(p) .$$

Thus, $L_x\left(v + \epsilon(p)\right) = L_x(v) + \epsilon(p)$, and, as there is only one such holomorphic map of C^k onto itself, it follows that $L_x(v) \equiv v$ for all $x \in M^*$. So we have shown that $G(v \cdot x) \equiv v + G(x)$. Since for any $p \in Z^{2k}$, $G\left(x \cdot f_*^{b_0}(p)\right) = G\left(\epsilon(p) \cdot x\right) = \epsilon(p) + G(x)$ the composite $e \circ G: M^* \to T$ induces a holomorphic $g: B \to T$ such that $g(tb) = tg(b)$ for all $t \in T$, $b \in B$. The global holomorphic section of $(T, B) \to W$ is then given by $g^{-1}(1)$.

Note that since $\phi\left(f_*^{b_0}(p)\right) \equiv \epsilon(p)$ the composite $e \bullet \phi: \pi \to T$ will also induce a crossed-homomorphism $\eta: N \to T$ such that

$$\alpha_*\left(g(b\alpha)\right) \equiv g(b)\eta(\alpha).$$

We may use this to define an action $\left(g^{-1}(1), N\right)$ which is equivariantly equivalent to (W, N). If $b \in g^{-1}(1)$ then $\alpha_*\left(g(b\alpha)\right) = \eta(\alpha)$ and thus $\alpha_*^{-1}\left(\eta(\alpha)^{-1}\right) \cdot b\alpha \in g^{-1}(1)$ also. But $1 = \eta(\alpha^{-1} \cdot \alpha)$ $= \eta(\alpha^{-1})\alpha_*^{-1}\left(\eta(\alpha)\right)$, so we let $b \circ \alpha = \eta(\alpha^{-1}) \cdot b\alpha$ for all $b \in g^{-1}(1)$, $\in N$. Observe that

$$\left(\eta(\alpha^{-1}) \cdot b\alpha\right) \circ \beta = \eta(\beta^{-1})\left(\left(\eta(\alpha^{-1}) \cdot b\alpha\right)\beta\right) = \left(\eta(\beta^{-1})\beta_*^{-1}\left(\eta(\alpha^{-1})\right)\right) b\alpha\beta = b \circ \left((\alpha\beta)\right).$$

We see that $\left(T, T \times g^{-1}(1)\right)$ is equivalent to (T, B) by $(t, b) \to tb$. If $\left(T \times g^{-1}(1), N\right)$ is given by $(t, b)\alpha = \left(\alpha_*^{-1}(t)\eta(\alpha^{-1})^{-1}, b \circ \alpha\right)$ then $\left(T, T \times g^{-1}(1), N\right) \to (T, B, N)$ becomes a $T - N$ equivariant equivalence. Now $m(\alpha) = \eta(\alpha)^{-1}$ is still a crossed-homomorphism so that we have shown that $\tau \in H^1(N, \mathcal{J})$ lies in the image of $H^1_\Phi(N; T) \to H^1(N; \mathcal{J})$ if $M_\tau \to V$ is a holomorphically injec-

tive Seifert fibering. It then follows that $\delta(\tau) \in H^2(N; \mathfrak{Z}^{2k})$ lies in the image of $H^2_{\Phi}(N; Z^{2k})$, and has finite order.

(10.5) Theorem: Let $\tau \in H^1(N; \mathfrak{J})$ be an element for which $\delta(\tau)$ is a Bieberbach class. Then τ lies in the image of $H^1_{\Phi}(N; T) \to H^1(N; \mathfrak{J})$ if and only if $M_\tau \to V$ is a holomorphically injective Seifert fibration.

Proof: We have already proved sufficiency. Suppose that τ is represented by a crossed-homomorphism $m: N \to T$. Then $(T \times W, N)$ is $(t, w)\alpha = \left(\alpha_*^{-1}(t) m(\alpha^{-1}), w\alpha\right)$. We must construct the fundamental group of M_τ together with its action on $C^k \times W$, the universal covering space of M_τ. We can choose a function $M: N \to C^k$ such that $e\left(M(\alpha)\right) = m(\alpha)$ for all $\alpha \in N$ and $M(e) = 0$. We should have $\pi = Z^{2k} \times N$ with $(p, \alpha)(q, \beta) = \left(p + \alpha_*(q) + c(\alpha, \beta), \alpha\beta\right)$ where $c(\alpha, \beta) \in Z^{2k}$ is the non-abelian extension cocycle. The action $(C^k \times W, \pi)$ must have the form

$$(v, w) \cdot (p, \alpha) = \left(\alpha_*^{-1}(v) + \alpha_*^{-1}\left(\epsilon(p)\right) + M(\alpha^{-1}), w\alpha\right) .$$

The composition rule is satisfied, however, if and only if

$$\beta_*^{-1}\left(M(\alpha^{-1})\right) + M(\beta^{-1}) = \beta_*^{-1}\alpha_*^{-1}\left(\epsilon\left(c(\alpha, \beta)\right)\right) + M(\beta^{-1}\alpha^{-1})$$

or equivalently

$$\epsilon\left(c(\alpha, \beta)\right) = \alpha_* M(\alpha^{-1}) - \alpha_*\beta_* M(\beta^{-1}\alpha^{-1}) + \alpha_*\beta_* M(\beta^{-1}) .$$

Since $\alpha_* m(\alpha^{-1}) \alpha_*\beta_*\left(m(\beta^{-1})\right) = m(\alpha)^{-1} \alpha_*\left(m(\beta)^{-1}\right) = m(\alpha\beta)^{-1} = \alpha_*\beta_* m(\beta^{-1}\alpha^{-1})$ there is for each $\alpha, \beta \in N \times N$ a unique $c(\alpha, \beta) \in Z^{2k}$ for which the required relation is satisfied.

Now let $G(v, w) = v$, then $\alpha_* G\left(\alpha_*^{-1}(v) + \alpha_*^{-1}\left(\epsilon(p)\right) + M(\alpha^{-1}), w\alpha\right) - G(v, w) = \epsilon(p) + \alpha_* M(\alpha^{-1})$. Thus $\phi: \pi \to C^k$ given by $\phi(p, \alpha) = \epsilon(p) + \alpha_* M(\alpha^{-1})$ is a crossed-homomorphism defining an element of $h^{1,0}(M_\tau, \mathcal{J})$. Under $F^*: h^{1,0}(M_\tau, \mathcal{J}) \to \mathrm{Hom}_c(C^k, C^k)^\pi$ the image of the element is the identity matrix. If L commutes with all α_* then ϕ is replaced with $L \circ \phi$ and G with $L \circ G$. Thus $F^*: h^{1,0}(M_\tau, \mathcal{J}) \to \mathrm{Hom}_c(C^k, C^k)^\pi$ is an epimorphism if τ lies in the image of $H^1_{\Phi}(N; C^k) \to H^1(N; \mathfrak{J})$.

Of course (10.5) applies to holomorphic actions of a complex toral group (T, M). In fact James Carrell shows that if M carries a Kähler metric, then the action is holomorphically injective. He uses (10.5) then to see how actions on Kähler manifolds arise.

11. An Example

According to Hollman [20], an effective holomorphic action $(T\ M)$ admits at every point of M a local holomorphic slice. Thus, as we mentioned, (7.3) is also valid in the analytic case. Now by default a principal action is locally injective. If M is a closed aspherical manifold then every action is injective and, as mentioned above, every holomorphic action on a closed Kähler manifold is holomorphically injective. Thus it would seem appropriate to give some examples of holomorphic actions which are not locally injective. We need only modify the Calabi-Eckmann examples [8] by introducing non-trivial isotropy subgroups. The torus will be that defined by

$$0 \to Z \xrightarrow{\epsilon} C^* \longrightarrow T \to 0$$

where $\epsilon(n) = e^{-2\pi n}$.

Fix an ordered k-tuple of positive integers $(p_1, \ldots, p_k)$ with greatest common divisor equal to 1. We define $(C^*, C^k \setminus \{0\}, Z)$ as follows

$$\lambda(z_1, \ldots, z_k) = (\lambda^{p_1} z_1, \ldots, \lambda^{p_k} z_k)$$

$$v \cdot n = \epsilon(n) \cdot v\ .$$

Then $(\lambda v) \cdot n = \lambda(v \cdot n) = \lambda\epsilon(n)(v)$, and so with $H(p_1, \ldots, p_k) = (C^k \setminus \{0\})/Z$ we have induced a holomorphic $\left(T, H(p_1, \ldots, p_k)\right)$. The action is principal if and only if $p_1 = \cdots = p_k = 1$. At each point $x \in H(p_1, \ldots, p_k)$ the induced homomorphism $\pi_1(T) \to \pi_1\left(H(\ ), x\right)$ is an epimorphism and hence the action fails to be locally injective unless it is principal. Of course $H(p_1, \ldots, p_k)$ is topologically $S^1 \times S^{2k-1}$.

Now choose an integer $j > 0$ and let $H(j)$ correspond to the j-tuple with every entry equal to 1. On the product $H(p_1, \ldots, p_k) \times H(j)$ there are two actions of T:

$$t(x, y) = (tx, t^{-1}y)$$

$$t(x, y) = (tx, y)$$

The first action is principal, so let M be the quotient of the product by this principal action. The second action then induces (T, M). Topologically, M is $S^{2k-1} \times S^{2j-1}$. Furthermore, the isotropy subgroups of (T, M) are the same as those of $\left(T, H(p_1 \ldots p_k)\right)$.

12. The Continuous Case

(12.1) In this section we shall discuss the continuous case as well as some equivariant topological reduction theorems which have important analogues when the objects involved have additional structure.

In section 7 we saw that replacing holomorphic (W, N) by a properly discontinuous group of diffeomorphisms placed us in the smooth category. For reasonable topological spaces W (those that are path connected, paracompact and having the homotopy type of a CW-complex or those that are paracompact, locally compact, path and locally path connected, and semi 1-connected spaces), the group of homeomorphisms N need only be properly discontinuous for the entire theory we have developed to carry through. Once again one considers a real k-torus

$$0 \longrightarrow Z^k \longrightarrow R^k \longrightarrow T \longrightarrow 0$$

and a homomorphism $\phi: N \longrightarrow GL(k, Z)$. We use <u>continuous</u> maps to define the sheaf with operators $\mathcal{J}_{R_c} \longrightarrow W$. For $j > 0$, $\delta: H^j(N; \mathcal{J}_{R_c}) \xrightarrow[\cong]{} H^{j+1}(N; \mathfrak{Z}^k)$, since $h^0_{R_c} \longrightarrow V$ is a fine sheaf. Similarly, $H^j_\phi\left(N; \mathrm{map}_{R_c}(W, R^k)\right) = 0$ for all $j > 0$ and the theory proceeds exactly as in the smooth case. There are certainly some advantages in the continuous case since one may wish to consider geometrically defined spaces with automorphisms (W, N) which are not smooth.

In both the smooth and continuous cases we have not considered the most general type of singular fibering that could arise when the generic fiber is a quotient of a torus by a freely acting finite group. Of course, what is missing is complete knowledge of the group of diffeomorphisms and homeomorphisms of the torus. We have replaced this lack of knowledge by the homomorphism $\phi: N \longrightarrow \mathrm{Aut}\, T$. Of course this is a reasonable working assumption for T itself is a subgroup of the connected component of the identity of the homeomorphism group of T and $\mathrm{Aut}\, T$ certainly is a subgroup of the group of path components. However, in treating singular fiberings which arise from actions, we, by assuming ϕ trivial, treat the most general possibility that may arise in this manner.

In 6.3, 6.4 and 6.5 we have discussed the special role played by holomorphic actions (kernel ϕ) in the general holomorphic Seifert fiberings. The key result needed was a theorem of E. Cartan. The real analogue of this result has been proved by A. Selberg, see [5 ; 2.3]. Hence 6-3 - 6.5 are valid in the continuous and smooth cases also. In particular, where $k = 1$, the Corollary 6.5 has a particularly nice form:

If τ represents a local action, which fails to be an action, then the Seifert fibre space M_τ has a double covering $M_{'\tau}$ and an action of the circle which projects to the local action.

Proof: $M_{'\tau}$ is nothing but the covering space induced by the double covering $W/K \longrightarrow W/N$ where $K = \text{kernel } \phi$, $\phi: N \rightarrow \mathrm{Aut}(Z) = Z_2$. Thus, if (S^1, B, N) is the coordinate bundle with operators N so that $M_\tau = B/N$, then $M_{'\tau} = (S^1, B/K)$.

In section 5, we discussed $\dim_c W = 1$. Here, everything is reduced to a properly discontinuous group of automorphisms operating on a simply connected 2-manifold. Since every orientation preserving smooth (respectively; continuous) properly discontinuous action is smoothly (respectively; continuously) equivalent to a holomorphic action we see that any even dimensional smooth (respectively; continuous) Seifert fibre space arising from orientation preserving homeomorphisms, is equivalent to a holomorphic one.

(12.2) Since a special role is played by the Seifert fibre spaces which are actions we would now like to discuss a procedure which often enables one to reduce the complexity of an action of a torus. This procedure is exploited in [15] from a different point of view.

Let (T, X) be a topological action on a space for which $H_1(X;Z)$ is finitely generated and X is locally compact and possesses the usual desirable local properties. We shall assume that (T, X) has only *finite stability groups*. We may consider the Leray spectral sequence of the orbit map $p: X \longrightarrow X/T$, where we use rational coefficient groups. With these coefficients the orbit map behaves like a principal fibre bundle map and in particular the Leray sheaf is constant. If we consider the terms of low degree from the spectral sequence we have

$$0 \rightarrow H^1(X/T;Q) \xrightarrow{p^*} H^1(X;Q) \xrightarrow{i^*} H^0\big(X/T; H^1(T;Q)\big) \xrightarrow{\delta} H^2(X/T;Q) \xrightarrow{p^*} H^2(X;Q) .$$

The homomorphism i^* may be identified with

$$H^1(X;Q) \xrightarrow{i^*} H^1(T/T_x;Q) \xrightarrow{(f^x)^*} H^1(T;Q) .$$

The composite above is the dual to the

$$f^x_*: H_1(T;Q) \longrightarrow H_1(X;Q)$$

considered in the fibering theorem. (We may use singular homology and cohomology here in dimension one since our local assumptions guarantee equivalence of the usual theories in

this low dimension.) The image of f_*^x is a vector space of dimension r. Consequently,

$$f_*^x : H_1(T, Z) \longrightarrow H_1(X; Z)$$

has rank r. We may find a splitting $T = T^k = T^r \times T^{k-r}$, where $f_*^x\big|_{T^r}$ is a monomorphism and $f_*^x\big|_{T^{k-r}}$ has finite image. Thus by the fibering theorem [14] the action restricted to the r-dimensional subtorus (T^r, X) fibers equivariantly over the torus $(T^r, T^r/\Delta)$ where Δ is a finite abelian subgroup of T^r. In fact, we have a finite abelian covering and equivariant maps:

$$\begin{tikzcd}
(T^r, T^r, \Delta) \arrow[d, "/\Delta"] & & (T^r, T^r \times Y, \Delta) \arrow[ll, "/Y"'] \arrow[r, "/T^r"] \arrow[d, "/\Delta"] & (Y, \Delta) \arrow[d, "/\Delta"] \\
(T^r, T^r/\Delta) & (T^k, X) \arrow[l, "/Y"'] = & (T^r, T^r \times_\Delta Y) \arrow[r, "/T^r"] & Y/\Delta
\end{tikzcd}$$

We shall now show that the entire action of T^k may be lifted to the abelian covering $T^r \times Y$. Let

$$H = \operatorname{im}(f_\#^x), \qquad f_\#^x : \pi_1(T^k, e) \longrightarrow \pi_1(X, x) .$$

We wish to show $H \subseteq \pi_1(T^k) \times \pi_1(Y) \subseteq \pi_1(X, x)$. We may assume that Y is path connected without any loss of generality.

Consider the commutative diagram

$$\begin{tikzcd}
& 1 \arrow[d] & \\
& \pi_1(T^r) \times \pi_1(Y) \arrow[d] & \\
\pi_1(T^k) \arrow[r, "f_\#^x"] & \pi_1(X) \arrow[r] \arrow[d, "j"] & H_1(X) \arrow[d] \\
& \Delta \arrow[d] & H_1(T^r/\Delta) = \pi_1(T^r/\Delta) \arrow[l] \\
& 1 &
\end{tikzcd}$$

In the composition $j \circ f_\#^x$, we may also consider the equivalent composite

$$\pi_1(T^k) \longrightarrow \pi_1(X) \longrightarrow H_1(X) \longrightarrow H_1(T^r/\Delta) \longrightarrow \Delta \quad .$$

Clearly, H lies in the kernel of j. Consequently, we may lift the action (T^k, X) to $(T^{r-k} \times T^r, T^r \times Y, \Delta)$ where Δ commutes with T^k. We may then consider the induced action $(T^{k-r}, Y, \Delta) = \left(T^{k-r}, (T^r \times Y)/T^r, \Delta\right)$. A calculation shows that the question as to whether (T^{k-r}, Y) arises from a Bieberbach class is equivalent to whether or not the restricted action (T^{k-r}, X) arises from a Bieberbach class.

We recall that the action (T, X) is *homologically injective* if the homomorphism $H_1(T, e; Z) \xrightarrow{f_*^x} H_1(X, x; Z)$ has trivial kernel. In the above, (T^r, X) is homologically injective while no connected non-trivial subgroup of T^{k-r} is homologically injective on X or on X/T^r. We shall summarize the discussion in the following

(12.3) *Reduction Theorem: Let (T^k, X) be an action with only finite stability groups. We assume that X has the usual local properties and that $H_1(X; Z)$ is finitely generated. Let r = rank $\operatorname{im}(f_*^x)$, where $f_*^x : H_1(T^k; Z) \longrightarrow H_1(X; Z)$ is the evaluation map on the first homology. Then there exists a decomposition $T^k = T^r \times T^{r-k}$ and (T^r, X) fibers equivariantly over $(T^r, T^r/\Delta)$. On the fibre Y, which can be chosen path connected, there is naturally induced an action of $T^{r-k} \times \Delta$ and T^{r-k} acts with only finite stability groups.*

The induced action (T^{r-k}, Y) is locally injective, injective or homologically injective, respectively, if and only if the same is true of the restricted action (T^{r-k}, X).

Furthermore, if dimension $H^2(X/T^k; Q) = s < k$, then $r \geq k - s$.

It remains to prove the homologically injective statement of the second paragraph. This will follow from the next lemma.

(12.4) *Lemma: Let (T^k, X') be a covering action of the injective action (T^k, X) and assume the covering $X' \to X$ is regular and of finite index. Then, (T^k, X') is homologically injective if and only if (T^k, X) is.*

Proof: Let $N = \pi_1(X, x)/\operatorname{im} f_*^x$, $N' = \pi_1(X', x')/\operatorname{im} f_*^{x'}$, $f_*^x : \pi_1(T^k, e) \to \pi_1(X, x)$. Let $\eta : Z^k \to Q^k$, and $i : N' \to N$ be the natural inclusions. Consider the diagram

$$\begin{tikzcd} H^2(N;Z^k) \arrow[r, "\eta_*"] \arrow[d, "i_Z"] & H^2(N;Q^k) \arrow[d, "i_Q"] \\ H^2(N';Z^k) \arrow[r, "\eta_*"] & H^2(N';Q^k) . \end{tikzcd}$$

Since i_Q^* is a monomorphism, any class $b \in H^2(N; Z^k)$ for which $i_Z^*(b)$ is of finite order goes trivially by η_* into $H^2(N; Q^k)$. Hence b had finite order in $H^2(N; Z^k)$ and by 7.5 (T^k, X) is homologically injective. On the other hand, if (T^k, X) is homologically injective the composition $H_1(T^k) \xrightarrow{f_*^x} H_1(X') \longrightarrow H_1(X)$ is a monomorphism by naturality. Consequently, (T^k, X') is homologically injective.

We apply the lemma in the reduction theorem above. If any connected subgroup of T^{r-k} in $(T^{r-k}, T^r \times Y)$ is homologically injective, the same would be true of this subgroup for the restriction (T^{r-k}, X) which is impossible. But any non-trivial connected subgroup of T^{r-k} whose restriction to (T^{r-k}, Y) is homologically injective would yield the same for $(T^{r-k}, T^r \times Y)$

[Closely related to 12.4 is the following proposition. Suppose that (W, N) is a properly discontinuous action of a finitely generated group on a simply connected space. Suppose there is a normal subgroup L of finite index for which the action (W, L) is free. Let $\tau \in H^2(N; \mathfrak{Z}^k)$ be a Bieberbach class and (T^k, X_τ) the locally injective action. The action (T^k, X_τ) is homologically injective if and only if the coordinate bundle with operators $(T^k, B, N/L)$ covering (T^k, X_τ) has characteristic class in $H^2(B/T^k; Z^k)$ of finite order.

Proof: If (T^k, X_τ) is homologically injective then $H^1(B; Z^k) \rightarrow H^1(T^k; Z^k) \xrightarrow{d} H^2(B/T^k; Z^k)$ is exact and the image of d is finite. Conversely, if image d is finite, then $H^1(B; Q^k) \rightarrow H^1(T^k; Q)$ is onto and hence (T^k, B) is injective. The lemma then implies that (T^k, X) is homologically injective. As a consequence when $H^2(B/T^k; Z) = H^2(W/L; Z)$ is free, then homological injectivity of (T^k, X) means that every (T^k, B), covering actions of finite index which are principal, must be trivial bundles. In particular, if N is finite it says that homological injectivity means injectivity. These facts are relevant to the examples in § 5, 7, 9 and those which follow.]

Another immediate consequence of 12.4 is a topological version of Calabi's reduction theorem for flat manifolds, see [14 ; 7.1]. The proof for the theorem in the reference cited is marred by several misprints. However, the argument is essentially the same as 12.4. The proof given above can be considered a correction of the misprints.

Calabi's theorem states essentially that if k is the rank of the center of $\pi_1(M,x)$ where M is a closed flat manifold, then the first betti number is k and M has a k-dimensional torus group of isometries which is homologically injective. The fibre Y is a flat manifold and $M = T^k \times_\Delta Y$, where Δ is a finite abelian group of isometries of Y.

Calabi has also conjectured a reduction theorem for closed Kähler manifolds [7]. Matsushima's result [26], mentioned in §7, is a proof of this conjecture for projective algebraic varieties. We shall now prove a topological version of Calabi's conjecture. This says that (T,M), an action of a torus on a closed homologically Kählerian manifold, is homologically injective if and only if all the stability groups are finite. In particular, using 8.4, a holomorphic (T,M) can always be deformed so that it holomorphically fibers when M admits a Kähler structure. This is a topological version of Carrell's result. It circumvents the material of § 10 but it does not yield the very refined form of the results of section 10.

We shall follow A. Borel (Seminar on Transformation Groups, Annals of Math. Study Chapter XII) and call a closed connected (cohomology) manifold M homologically Kählerian if there exists a class $Q \in H^2(M;K)$ such that

$$H^s(M;K) \xrightarrow{\cup Q^{n-s}} H^{2n-s}(M;K)$$

is an isomorphism, $s = 0,1,2,\ldots,n$, where K is some field of characteristic 0. (A compact complex manifold which admits a Kahler metric is homologically Kählerian.)

(12.6) Theorem: Let (T,M) be an action on a homologically Kählerian (cohomology) manifold. Then (T,M) has finite stability groups if and only if (T,M) fibers equivariantly over T.

We obtain the result first for the circle and then extend to the product of circles. Our first lemma is a variant of A. Borel's theorem 6.2 of Chap. XII of [4].

(12.7) Lemma: Let (S^1,M) be an action on a homologically Kählerian manifold. Then $F(S^1,M) = \emptyset$ if and only if $d^2(H^1(M;K)) \neq 0$ in $H^2(B_{S^1};K)$.

Proof: We look at the spectral sequence associated with the fibering of the Borel space

$$M_{S^1} = E \times_{S^1} M \xrightarrow[\pi_2]{/M} B_{S^1} .$$

where E is the universal S^1-bundle over the universal classifying space B_{S^1} for the circle.

A theorem of Blanchard [3; Theorem II, 1.2] states that if there is a fibering whose total space is compact and a typical fiber is a connected compact homologically Kählerian manifold, then in the Leray spectral sequence, with $E_2 = H^*(\text{base};K) \otimes H^*(\text{fibre};K)$ and $d^2\left(H^1(\text{fibre}, K)\right) = 0$, the fibre is totally non-homologous to 0 in the total space. If $d^2 = 0$, then Borel 3.4 of [4 ; Chap. XII] yields that the fixed point set F must be nonempty. If, on the other hand, $d^2 \neq 0$ and $F \neq 0$, then $\pi_2^*: H^2(B_{S^1};K) \to H^1(M_{S^1}, K)$ is injective. The kernel of π_2^* contains $d^2\left(H^1(M,K)\right)$ according to A. Borel (Annals of Math. 57, 1953, 115-207, § 5). But this is zero which yields a contradiction.

We now assume (S^1, M) is without fixed points. We examine the Borel diagram:

$$\begin{tikzcd}
M \arrow[d, "/S^1"] & M \times E \arrow[l] \arrow[r] \arrow[d, "'\pi"', "/S^1"] & E \arrow[d, "/S^1"] \\
M/S^1 & M \times_{S^1} E \arrow[l, "\pi_1"'] \arrow[r, "/M", "\pi_2"'] & B_{S^1}
\end{tikzcd}$$

Since we are using the field K as coefficients, the map $M/S^1 \longrightarrow M/S^1$ is a Vietoris map and we may homologically identify $M \longrightarrow M/S^1$ with the fibering $M \times E \longrightarrow M \times_{S^1} E$. We consider the exact sequence of terms of low degree for the two fiberings $M \times E \longrightarrow M_{S^1}$, and $M_{S^1} \longrightarrow B_{S^1}$. Naturality enables us to construct the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(M/S^1, K) \arrow[r, "'\pi^*"'] \arrow[d, "\cong"', "\pi_1^*"] & H^1(M;K) \arrow[r, "\text{edge}", "j^*"'] \arrow[d, "\cong"'] & H^1(S^1, K) \arrow[r, "d^2"] & H^2(M/S^1;K) \\
0 = H^1(B_{S^1};Q) \arrow[r, "\text{edge}"] & H^1(M_{S^1};K) \arrow[r, "i^*", "\text{edge}"'] & H^1(M;K) \arrow[r, "d^2"] & H^2(B_{S^1};Q) \arrow[r, "\text{edge}"'] & H^2(M/S^1;K)
\end{tikzcd}$$

From the fact that i^* is not onto (since $d^2(\ )$ is not 0), it follows that $'\pi^*$ is not onto, and hence j^* must be surjective. Of course the homomorphism j^* can be factored:

$$\begin{tikzcd} H^1(M,K) \arrow[r, "j^*"] \arrow[dr, "f^x_*"'] & H^1(S^1/S^1_x;K) \arrow[d, "\cong"] \\ & H^1(S^1;K) \end{tikzcd} .$$

Consequently, we have that $F = \emptyset$ implies that $f^x_{\#}: H_1(S^1; Z) \longrightarrow H_1(M; Z)$ is a monomorphism and hence (S^1, M) fibers equivariantly over $(S^1, S^1/\Delta)$ for some finite $\Delta \subset S^1$.

We extend now to actions of the torus. We may split the torus $T^k = T^j \times T^{k-j}$ so that $F(T^j, X) \neq \emptyset$ and T^{k-j} acts with only finite stability groups. The group $H^2(B_{T^k}; K)$ splits naturally into $H^2(B_{S^1_1}; K) \oplus \ldots \oplus H^2(B_{S^1_k}; K)$. We may consider the coboundary d^2 of the lemma to be $d^2_1 \oplus \quad \oplus d^2_k$ by restricting the action of T^k to the compoments of T^k, $(S^1_1 \times \ldots \times S^1_j) \times (S^1_{j+1} \times \ldots \times S^1_k)$. The proof of the lemma now carries over.

We now assume that (T^k, M) has only finite stability groups. We replace the circle group by T^k in the commutative diagram relating the terms of low degree of the two spectral sequences. Since d^2 is surjective, j^* must be surjective and that completes the proof.

We regard this result as a type of reduction theorem. It states that a toral action (T^k, M) on a homologically Kählerian manifold is determined by an action of a finite abelian group $\Delta \subset T^k$ acting on a closed submanifold $Y \subset M$, the fibre. We now observe that it is often possible to check that Y is also homologically Kählerian. Thus, in this case, we could repeat this process of simplification of M by fibering Y over a torus if there were a toral action on Y with only finite stability groups. Of course, in section 10, when dealing with an actual Kähler manifold, the analytically embedded fibre Y is necessarily a Kähler submanifold. That is if we assume that (T, M) is holomorphic with M Kähler, then when deforming (T, M) to a class of finite order, M remains Kähler and the resulting holomorphic action holomorphically fibers. <u>Thus on the fibre Y there is the finite abelian group Δ of isometries and holomorphically (T, M) is $(T, T \times_\Delta Y)$.</u>

(12.8) We now wish to combine the two theorems in this section for some more information concerning the case treated in § 5.

When an action (T^k, M) is homologically injective, then $f^x_*: H^1(X; K) \longrightarrow H^1(T^k; K)$ is onto, and $H^1(T^k; K)$ generates the cohomology of T^k. Thus we get $E_2^{i,j} = E_\infty^{i,j}$ and,

$$\sum_{i+j=n} H^i(X/T^k;K) \otimes H^j(T^k;K) \xrightarrow[\phi]{\cong} H^n(X;K) ,$$

where the isomorphism ϕ is given by $\phi(a \otimes b) = p^*(a) \cup \theta(b)$. Here $p: X \to X/T^k$ is the orbit map and θ is a splitting so that $f_*^x \circ \theta$:

$$H^n(T^k, K) \longrightarrow H^n(X;K) \xrightarrow{f_*^x} H^n(T^k/T^k_x;K)$$

is the identity. In particular, the betti numbers of X are determined if those of X/T^k are known. For example, if k is even, and the odd betti numbers of X/T^k are even, then the odd betti numbers of X are even.

Our special case of interest arises when $H^2(X/T^k;K)$ is 0 or 1-dimensional. Then the image of $f_*^x: H^1(X;K) \to H^2(T^k;K)$ is the kernel of $d: H^1(T^k, K) \to H^2(X/T^k;K)$ is of rank k or $k-1$. Clearly, we may topologically recapture Lemma 8.1. More generally, if X/T^k is a closed 2-manifold and k is even, then $b_1(X)$ is even if and only if (T^k, X) is homologically injective.

In order to formulate our remarks on actions on manifolds of codimension 2 it is necessary to examine the 3-dimensional manifolds first. The actions of the circle on 3-manifolds have all been classified as Seifert manifolds and the succeeding lemmas can be deduced directly from this classification [33], [32], [29] and [27]. However, to illustrate application of our methods we shall deduce as much as we can from our present point of view and refer to the classification only in the treatment of actions which fail to be locally injective in (12.11) and (12.12). In the following the running hypothesis is that (S^1, M^3) is an effective action without fixed points on a connected 3-manifold without boundary.

(12.9) Lemma: The following are equivalent:

1. (S^1, M^3) is injective
2. $\pi_1(M^3)$ is infinite
3. M^3 is covered by $\mathbb{R}^3$ or $S^2 \times S^1$.

To determine those that are homologically injective we have:

(12.10) Lemma: (S^1, M^3) is homologically injective, if and only if,

1. M^3 is non-compact or is nonorientable, or
2. if M^3 is compact and orientable, then the first betti number is odd.

From 12.9 we see that (S^1, M^3) can fail to be injective if and only if $\pi_1(M^3)$ is finite. Thus M^3 must be closed and orientable. From the proof of 12.9, M^3/S^1 will necessarily have to be a 2-sphere. Yet some of these actions may even fail to be locally injective. We now wish to characterize those which are still locally injective. We shall break it up into the abelian and non-abelian cases.

(12.11) Lemma: $\pi_1(M^3)$ is finite and non-abelian if and only if M^3/S^1 is a 2-sphere with exactly three singular orbits Z_{α_1}, Z_{α_2}, Z_{α_3} so that $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 > 1$. In this case (S^1, M^3) is necessarily locally injective.

(12.12) Lemma: $\pi_1(M^3)$ is finite and abelian if and only if (S^1, M^3) is topologically equivalent to a "linear" action on a lens space $(\neq S^2 \times S^1)$. The orbit space is necessarily a 2-sphere with fewer than 3 singular orbits. The action is locally injective if and only if there are no singular orbits or exactly two singular orbits with the same stability group.

Proofs of Lemmas 12.9 - 12.12. We shall, for convenience, assume that $H_1(M^3; Z)$ is finitely generated and that there are at most a finite number of distinct isotropy subgroups although neither assumption is really necessary to carry out a proof. From the topological slice theorem we see that the orbit space is a 2-manifold. The 2-manifold has empty boundary unless there are orbits with Z_2 stability groups which reverse the orientation of a slice. In this latter case, these orbits project to the boundary and the original 3-manifold is necessarily non-orientable. The orientability of the orbit space is the same as that of M^3 unless there are Z_2-type stability groups which locally reverse orientation. In this latter case the orientability of the orbit space is determined by the orientability of M^3 with these exceptional orbits removed.

Let F denote the smallest subgroup of S^1 containing all the stability groups. Then the orbit map

$$(S^1/F, M^3/F) \longrightarrow \left((M^3/F)\big/(S^1/F) = M^3/S^1\right)$$

is a principal fibering. If M^3/S^1 is not the 2-sphere or the projective plane, then it is a $K(\pi, 1)$ and hence the free action $(S^1/F, M^3/F)$ is injective. Naturality then implies the action of (S^1, M^3) is injective. In case M^3/S^1 is the projective plane then the two principal circle bundles over it are homologically injective actions. Thus injectivity could only fail when M^3 is closed, orientable and the orbit space is S^2.

We shall now show that $\operatorname{im} f_*^x$ is a finite subgroup of $\pi_1(M^3)$ if and only if $\pi_1(M^3)$ is a finite group. Suppose that $\operatorname{im} f_*^x$ is finite. Choose a homomorphism $j:{}'S \to S^1$ so that $\operatorname{im}(j_*)$ is kernel f_*^x, when $j_*: \pi_1('S^1, 1) \to \pi_1(S^1, 1)$. We may now lift $('S^1, M^3)$ to an action on the universal covering $\widetilde{M}^3$. We perform a construction analogous to above and form $('S^1/'F, M^3/'F)$. This is a free action on a simply connected space and the quotient $\widetilde{M}^3/'S^1$ is a simply connected 2-manifold. It can only be S^2 for otherwise $\pi_1(\widetilde{M}^3/'F)$ could not be trivial. Thus $\widetilde{M}^3/'F$ is the 3-sphere and $\widetilde{M}^3$ is a closed simply connected 3-manifold. Hence, $\pi_1(M^3)$ is finite (and really covered by the 3-sphere). This yields the first equivalences of 12.9.

We now examine the M^3 which are not $K(\pi,1)$'s, but still have infinite fundamental group. Since (S^1, M^3) is injective, we may lift to the "splitting action", $(S^1, M^3_{(\operatorname{im} f_*^x)})$ by 7.4. Since $(M^3_{(\operatorname{im} f_*^x)}/S^1)$ is simply connected and not a $K(\pi,1)$ it must be S^2. (This yields the third equivalence of 12.9.) Thus, (S^1, M^3) has a finite covering by $(S^1, S^1 \times S^2, \Delta)$ where $\Delta = \pi_1(M^3)/\operatorname{im} f_*^x$ which must operate effectively on S^2. Thus, as a Bieberbach class, (S^1, M^3) may be regarded as an element of $H^2(\Delta; Z)$. Since Δ is finite, the Bieberbach class must be of finite order. Consequently, $(S^1, M^3) = (S^1, S^1 \times_{\Delta'} S^2)$ or $(S^1, S^1 \times_{\Delta'} P^2)$. It can also be regarded as a Bieberbach class of the cyclic group Δ'. Each of these are S^2 or P^2 bundles over S^1/Δ' with structure group Δ'. The action (S^1, M^3) has exactly 0, 1, 2 singular orbits and/or a circle of exceptional orbits; 1 occurs only when M is not orientable. Clearly, $M^3 = S^2 \times S^1$ where orientable; otherwise we have $P^2 \times S^1$ or the non-orientable 2-sphere bundle over the circle. The orbit spaces are S^2, P_2 or the disk.

We have seen that all the actions which are injective are homologically injective if M^3 is not a $K(\pi,1)$. If (S^1, M^3) is injective and M^3 is a $K(\pi,1)$, then the action can be described as a Bieberbach class in $H^2(N; Z)$, where $N = \pi_1(M,x)/\operatorname{im} f_*^x$ and N operates properly discontinuously on $\mathbb{R}^2$. From $0 \to H^2(M^3/S^1; Z) \to H^2(N; Z) \to H^0(M^3/S^1; h^2) \to 0$ (from the 'E spectral sequence), we have that $H^2(N; Z)$ is of finite order whenever $H^2(M^3/S^1; Z)$ is of finite order. This is precisely the case whenever M^3 is not closed and orientable, and consequently, the action is homologically injective. In the closed orientable case, we may apply 12.8. This completes the proof of 12.10.

Let us now examine the locally injective case when $\pi_1(M^3)$ is finite. Since the lifted action $(S^1, M_{(\operatorname{im} f_*^x)})$ is free and has S^2 as a quotient space, the action is represented by a lens space principally fibered over the 2-sphere. The quotient $N = \pi_1(M)/\operatorname{im} f_*^x$ operates effectively on the 2-sphere and necessarily must be topologically equivalent to a <u>linear orientation preserving</u> action. Hence, if N is non-abelian (and orientation preserving),

there are exactly three distinct orbits where the stability group is not trivial. The stability groups, which are cyclic, have orders $\alpha_1, \alpha_2, \alpha_3$ where

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1 .$$

(All possibilities can occur.) If N is abelian, then the central extension

$$0 \longrightarrow \operatorname{im} f_*^x \longrightarrow \pi_1(M^3) \longrightarrow N \longrightarrow 1$$

is abelian as we shall now observe. Since N is abelian it must be cyclic, $Z_2 \oplus Z_2$ or e. Thus there are no or exactly two singular orbits. If N = e, then clearly (S^1, M^3) is a principal fibering over S^2 different from $S^1 \times S^2$. If $N \neq e$, then we have exactly 2 singular orbits where the stability groups are identical. Thus M^3 is the union of two solid tori neighborhoods of the singular orbits. These locally injective actions on lens spaces can be described as the action induced by Hopf action (principal S^1-action on the 3-sphere) on the quotient lens space $S^3/\pi_1(M^3) = M^3$. This action may not be effective but when made effective it or its inverse is the required action. To see this one observes that there is a minimal covering group $'S^1 = S^1$, which can be lifted to the 3-sphere. The covering transformations which are linear on the 3-sphere commute with the lifted action which is topologically the Hopf action. In the non-abelian locally injective case, we have the similar result that the action (ineffective) is induced from the covering Hopf action. What is not so clear in this case is that $\pi_1(M^3)$ can be regarded as a group of linear transformations of the 3-sphere. This is actually true but the argument is complicated.

The classification of the remaining non-locally injective actions could proceed as just mentioned above. One lifts a covering circle group to an effective action on a closed simply connected 3-manifold. One needs to prove that there are at most 2 singular orbits on the universal covering and so $\widetilde{M}^3$ would have to be the union of two solid tori sewn equivariantly together along the boundary and producing, topologically, a linear action on the 3-sphere. One would also need to know that on M^3 there are at most 2 singular orbits to identify M^3 as a lens space. It is precisely at this point that we proceed along the lines of the classification theorems [33], [32] and [27]. We use only a small part of this method.

If (S^1, M^3) has S^2 as an orbit space we may remove the interiors of invariant neighborhoods of the singular orbits. We obtain the product of a punctured 2-sphere with the circle. Now using the Van Kampen theorem we may, following Seifert [33], find a presentation of the fundamental group of M^3:

$$\pi_1(M^3) = \left\{q_i, h \,\middle|\, q_1 \cdots q_n h^{-b}, q_i^{\alpha_i} h^{\beta_i}, [q_i, h]\right\}, \qquad i = 1, 2 \ldots n .$$

Here the q_i are the generators of the fundamental group of cross-sectional curves to the invariant tubular neighborhoods and h represents the generator of the fundamental group of the principal orbit. The relative prime pairs (α_i, β_i), $0 < \beta_i < \alpha_i$ are the numbers, normalized, arising from the slice representation:

$$z \longrightarrow z\exp(2\pi\, \beta_i'/\alpha_i)\ , \qquad \beta_i'\beta_i \equiv 1(\alpha_i)$$

in a neighborhood of each singular orbit. The symbol b represents an arbitrary integer and is a certain obstruction to completing a cross-section over the punctured S^2 when the q_i are specified on the boundary. Clearly the group, (h), generated by h is in the center. Hence,

$$N = \pi_1(M)/(h) = \left\{ q_i \,\middle|\, q_1 q_2 \cdots q_n,\ q_i^{\alpha_i} \right\},$$

is a well known type of group. This group acts on the Euclidean plane with compact quotient if $n > 3$, and if $n = 3$, $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 \leq 1$. Thus N would have to be infinite. Otherwise N is finite and acts on the 2-sphere. Thus $\pi_1(M^3)$ is finite, if and only if N is finite. When $n \leq 2$, N is abelian and $\pi_1(M^3)$ is abelian, hence cyclic. If $n = 3$, N is non-abelian and $\pi_1(M^3)$ is non-abelian.

We now wish to show that the non-abelian is <u>necessarily</u> locally injective. Since (h) is in the center and since $\pi_1(M^3)/(h) = N$ is centerless, in the non-abelian case, we see that (h) is precisely the center. Furthermore (h) is precisely $\mathrm{im}(f_*^x)$ since it represents a principal orbit. We may lift our action to the lens space $(S^1, M^3_{(h)}, N)$. But the stability groups Z_{α_1}, Z_{α_2}, Z_{α_3} of (S^1, M^3) are represented by isomorphic stability groups of the action (S^2, N). This action is topologically linear and there are exactly 3 types of orbits with cyclic stability groups isomorphic to $(q_i^{\alpha_i})$. Thus, the homomorphism of 7.3

$$\eta_x : S^1_x \longrightarrow \pi_1(M^3)/(h) = N$$

is a monomorphism. Consequently, (S^1, M^3) <u>fails to be locally injective only when</u> M^3 <u>is a lens space</u> $\neq S^2 \times S^1$.

To see exactly when this arises we calculate the order of h in $\pi_1(M^3)$. It turns out to be g.c.d.(α_1, α_2) if there are two singular orbits and to be that of $\pi_1(M^3)$ if there is exactly one. Thus, (S^1, M^3) can be locally injective if and only if $\alpha_1 = \alpha_2$ or there are no singular orbits. This will complete the proof of all the lemmas after we calculate the order of h in $\pi_1(M^3)$.

This final computation, which can be done directly by examining carefully the sewing of the two solid tori, will be obtained from the following more general calculation. We shall

determine precisely when (S^1, M^3) is homologically injective in terms of the presentation of $\pi_1(M^3)$, or better yet in terms of the slice representations, and the obstruction class b. Obviously we need only consider M^3 when it is closed and oriented, otherwise it is always homologically injective. The Seifert presentation of $\pi_1(M^3)$ is then similar to that given when M^3/S^1 is S^2. We just need to add to the presentation the generators of $\pi_1(M^3/S^1)$ together with the induced relations:

$$\pi_1(M^3) = \left\{ a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_n, h \;\middle|\; [a_1, b_1] \cdots [a_g, b_g] q_1 \cdots q_n h^{-b}, \; [q_1, h], [a_j, h], [b_j, h], q_i^{\alpha_i} h^{\beta_i} \right\},$$

$$i = 1, \ldots, n, \; j = 1, \ldots, g.$$

Now we abelianize and obtain:

$$H_1(M^3; Z) = \begin{cases} Z_{2g} \oplus \text{Torsion} & \text{or} \\ Z_{2g} \oplus Z + \text{Torsion}. \end{cases}$$

The relations for the torsion are

$$\begin{array}{llll} q_1 + q_2 & \ldots + q_n & -bh & = 0 \\ \alpha_1 q_1 + 0 & \ldots + 0 & + \beta_1 h & = 0 \\ \vdots & & \vdots & \\ 0 + 0 & \ldots + \alpha_n q_n & + \beta_n h & = 0 \end{array}$$

We obtain the following relation matrix:

$$\begin{pmatrix} 1 & 1 & \ldots & 1 & -b \\ \alpha_1 & 0 & \ldots & 0 & \beta_1 \\ 0 & \alpha_2 & \ldots & 0 & \beta_2 \\ \vdots & & & & \\ 0 & 0 & \ldots & \alpha_n & \beta_n \end{pmatrix}.$$

From this we see that the determinant is the order of the torsion. Its value is

$$o(T) = (b\alpha_1\alpha_2\ldots\alpha_n)+(\beta_1\alpha_2\ldots\alpha_n)+(\alpha_1\beta_2\alpha_3\ldots\alpha_n)+\ldots+(\alpha_1\ldots\alpha_{n-1}\beta_n)\ .$$

If, following Orlik, Vogt and Zieschang, we put $A = \ell cm(\alpha_1, \ldots \alpha_n)$, and $\alpha_i' = A\alpha_i^{-1}$, then

$$o(h) = bA + \sum_{i=1}^{i=n} \alpha_i'\beta_i \quad ,$$

where $o(h)$ is the order of the element h in $H_1(M^3; Z)$. Clearly, (S^1, M^3) is homologically injective, if and only if $o(h) = 0$. This formula was obtained earlier by Orlik, Vogt and Zieschang [29] in a similar context. Now let us observe that

$$b + \frac{\beta_1}{\alpha_1} + \ldots + \frac{\beta_n}{\alpha_n} = \frac{o(T)}{\alpha_1\alpha_2\ldots\alpha_n} = \frac{o(h)}{A} \quad .$$

Hence, if $o(h) \neq 0$,

$$\frac{o(T)}{o(h)} = \frac{\alpha_1\alpha_2\ldots\alpha_n}{A} \quad .$$

<u>In particular, if</u> $n \leq 2$, <u>and</u> $g = 0$, <u>we are in the abelian case and</u> $o(T) = o(\pi_1(M^3))$. <u>Thus,</u>

$$\frac{o(\pi_1(M^3))}{o(h)} = \frac{\alpha_1\alpha_2}{A} = \text{q.c.d.}(\alpha_1, \alpha_2) \ .$$

In the next section, as well as in [16] we shall have use for the following observation.

(12.13) <u>Corollary</u>: <u>If</u> M^3 <u>is closed and oriented, then</u> (S^1, M^3) <u>is homologically injective if and only if</u>

$$b + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \ldots + \frac{\beta_n}{\alpha_n} = 0 \ .$$

<u>In particular, the sum of irreducible fractions</u>

$$\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \ldots + \frac{\beta_n}{\alpha_n}$$

<u>must be integral.</u>

The proof is an immediate consequence of the preceding formulae. If this sum is not integral, $N = \pi_1(M^3)/\mathrm{im}\, f^x_*$ never has a torsion-free normal subgroup with finite abelian quotient, cf. [15].

(12.14). Finally we should mention what happens if M^3 possesses non-empty boundary. We may still use the trick of dividing out the smallest subgroup containing all the isotropy subgroups. We obtain, then, a 2-manifold with non-empty boundary as an orbit space. Thus the principal fibering $(M^3/F) \longrightarrow (M^3/S^1)$ is homologically injective. Once again, (S^1, M^3) would then necessarily be homologically injective.

The local S^1 actions (in the sense of § 4) on 3-manifolds may now be described. From (6.5) and (12.1) they are all doubly covered by locally injective S^1-actions. This double covering can be different from the orientable double covering obtained in [27] .

We assume now that (T^k, M^{k+2}) is an effective action on a connected $(k+2)$-dimensional manifold with only finite stability groups. As in the preceding lemmas, we assume, for convenience, that there are only a finite number of distinct isotropy groups, $H_1(M^{k+2}; Z)$ is finitely generated and that M is boundaryless, but not necessarily compact. By the slice theorem M^{k+2}/T^k is a connected 2-manifold exactly as in the preceding lemmas. The rank $H^2(M^{k+2}/T^k; Z)$ is 1 if and only if M is both closed and orientable, otherwise the rank is 0. Consequently, Theorem (12.3) guarantees a reduction of the action (T^k, M^{k+2}) to $(T^k, T^k \times_{\Delta'} S)$ or $(T^{k-1} \times T^1, T^{k-1} \times_{\Delta} Y)$. In the general latter case the restricted action, (T^{k-1}, M^{k+2}), fibers equivariantly over $(T^{k-1}, T^{k-1}/\Delta)$ with connected fiber Y. Associated to this fibering is the induced action $(T^1 \times \Delta, Y) = (T^1, Y, \Delta)$.

(12.15) Theorem: Each associated action (T^1, Y, Δ) to (T^k, M^{k+2}) is an action of the circle on a connected 3-manifold without fixed points. Local injectivity, injectivity and homological injectivity of the associated induced actions are equivalent to the same for (T^k, M^{k+2}).

All locally injective actions on even dimensional orientable M^{k+2} are topologically equivalent to holomorphic actions. For closed orientable even dimensional M, homological injectivity is equivalent to M admitting a structure of a projective algebraic non-singular variety which, in turn, is equivalent to M admitting a Kähler structure.

Proof: Assuming only continuous actions it follows that (S^1, Y) is an action on an integral cohomology 3-manifold. The slice theorem immediately implies that Y is locally Euclidean. The Reduction Theorem 12.3 yields the first part of the theorem.

As we have already seen at the beginning of this section, every (T^k, M^{k+2}) which is locally injective (arises from a Bieberbach class associated with a principal coordinate bundle with operators) is topologically equivalent to a holomorphic action.

Where M^{k+2} is even dimensional, closed and orientable then M^{k+2} is homologically injective if and only if dimension $H_1(M^{k+2}; K)$ is even. Thus those that do not fiber equivariantly over T^k/Δ can not be Kähler manifolds. On the other hand those that do fiber are homeomorphic to projective algebraic varieties. Since the action is equivalent to a holomorphic action we may as well assume that (W, N), the induced action on $(B/T, \pi_1(M)/\mathrm{im}\, f_*^x)$, is a group of Kähler isometries on the simply connected Riemann surface B/T. Since the action topologically fibers we may deform the corresponding holomorphic action so that the resulting holomorphic action holomorphically fibers over T^k/Δ. This induces the holomorphic action on the Δ covering of M^{k+2}, $(T^k, T^k \times S, \Delta)$ where S is a closed Riemann surface and Δ is a group of isometries on S. Since S is algebraic, Δ can be regarded as algebraic. If we have chosen an algebraic torus then $(T^k, T^k \times_\Delta S)$ will be algebraic.

We may characterize the local injectivity, injectivity and homological injectivity of (T^k, M^{k+2}) in terms of the fundamental group and orbit space in the same manner that we employed for S^1-actions on 3-manifolds. Obviously homological injectivity will occur whenever the orbit space is not closed or not orientable. In these cases M^{k+2} is also a $K(\pi, 1)$. When M^{k+2} is closed and orientable, then injectivity can fail only when M^{k+2}/T^k is S^2. The rank of $H_1(M^{k+2}; Z)$ determines homological injectivity. We could then go to a presentation of $\pi_1(M^{k+2}, x)$ similar to that employed in the lemmas and deduce once again that having more than 3 singular orbits when M/T is the 2-sphere forces M to be a $K(\pi, 1)$. The analysis with fewer than 4 orbits proceeds analogously.

In conclusion we shall topologically characterize those actions of (T^k, M^{k+2}) for which $\pi_1(M^{k+2})$ is abelian. Simultaneously this will _characterize those actions which fail to be locally injective_. Let us assume that by the Reduction Theorem we have written (T^k, M^{k+2}) as $(T^{k-1}, T^{k-1} \times_\Delta Y)$ where the fiber Y is a 3-dimensional lens space where we allow $S^2 \times S^1$ and S^3. If $\Delta = Z_2 \oplus Z_2$ or Z_2 we shall say the reduction is _special_.

(12.16) _Lemma_: _If the reduction_ $(T^{k-1}, T^{k-1} \times_\Delta Y)$ _of an orientable_ (T^k, M^{k+2}) _is not special, then_ M^{k+2} _is homeomorphic to_ $T^{k-1} \times Y$.

Proof. We shall show that Δ is cyclic or $Z_2 \oplus Z_2$. When cyclic it can be embedded in some circle subgroup T_1 of T^{k-1}. We may write the fibering over T^{k-1}/Δ as a fibering over $T^{k-2} \times (T_1/\Delta)$ for some product decomposition $T^{k-2} \times T_1$ of T^{k-1}. As an action of

T^{k-1}, M^{k+2} is now $\left(T^{k-2}\times T_1,\ T^{k-2}\times(T_1\times_\Delta Y)\right)$. We then show, when $\Delta \neq Z_2$, that the action (Y,Δ) is embedded in a 2-torus action on Y. Thus, the homeomorphism, represented by the generator of Δ, is isotopic to the identity. Hence the lens space bundle $T_1 \times_\Delta Y$ over T_1/Δ is a product when we enlarge the structure group to the torus containing the Δ-action.

First, we observe that we may as well assume (T^k, M^{k+2}) is effective. Then it follows easily that the actions (Y,Δ) and $(Y/T^1,\Delta)$ are effective. Since Δ is abelian and Y/T^1 is the 2-sphere it follows that Δ is cyclic or $Z_2 \oplus Z_2$. The group Δ sends the circle orbits of Y into themselves. There are at most 2 orbits of Y for which $T^1_y \neq$ identity. These orbits can never project to free orbits of $(Y/T^1,\Delta)$, unless $\Delta = Z_2$. On S^2, Δ is topologically equivalent to an orientation preserving linear action. Let us assume that Δ is cyclic and not Z_2 and so the action of $\Delta \cong Z_n$ is equivalent to rotations about 2 fixed points p and q of period n. Over $S^2-\{p\cup q\}$ the action of Δ and of T^1 are free. The action of T^1 is therefore a product action. Since over $(S^2-\{p\cup q\})/Z_n$ the restriction of the action of T^1 on Y/Z_n is also free, this action also must be a product action. Consequently over $S^2-\{p\cup q\}$ the S^1-action is equivalent to

$$\left(T^1,\ T^1\times(S^1\times R^1),\ Z_n\right)\xrightarrow{/Z_n}\left(T^1,\ T^1\times_{Z_n}(S^1\ \ R^1)\right).$$

This means that over $S^2-\{p\cup q\}$ the action Δ is naturally embedded in a free torus action. Now we use the fact that the fixed point free circle actions on a lens space may be obtained by restricting the 2-dimensional toral actions on the lens space to the circle subgroups which do not appear as isotropy subgroups. See [28] for complete details. We choose a toral action (T,Y) on our given lens space and a circle subgroup $T^1\subset Y$ whose restriction yields (T^1,Y). The quotient Y/T is an arc and there exists a cross-section $Y/T \to Y$ to the orbit map. By projecting the section to $Y/T^1 = S^2$, we get a section of $(T/T^1, S^2) \longrightarrow Y/T$. Our action of Z_n on Y when restricted to the (possibly) two singular orbits is clearly embedded on T^1. On the rest of Y, Z_n is embedded in T. Since the product action of T^1 over $S^2-\{p\cup q\}$ extends to the desired toral action on Y, the given Z_n action on Y is embedded in a toral action because the extension must agree. Therefore the bundle $M^{k+2}\to T^{k-1}/\Delta$ is a product $(T^{k-1}/\Delta)\times Y$. Actually the proof shows that Δ can be embedded in a circle subgroup of a torus which acts on Y.

Another consequence of the above analysis is that the orbit space of (T^k, M^{k+2}) is a 2-sphere with at most 2 singular orbits. We presume in the remaining two cases which we have called special that the manifold is still a product $T^{k-1}\times Y$, however a more careful analysis of (Y,Δ) is probably necessary.

(12.17) Corollary: Let (T^k, M^{k+2}) be an effective action with only finite stability groups. Assume that M is orientable and $\pi_1(M)$ is abelian. Moreover, if also closed assume that not every reduction is special. Then, if the action is injective, M^{k+2} is homeomorphic to $T^k \times S$ where S is either the 2-sphere, torus, plane, or open annulus. If not injective, then M^{k+2} is homeomorphic to $T^{k-1} \times L$, where L is a lens space different from $S^2 \times S^1$.

Proof. If the action is injective, it is homologically injective since $\pi_1(M^{k+2})$ is assumed to be abelian. Since $\pi_1(S)$ must be abelian S must be either R^2, S^2, T^2 or $R^1 \times S^1$. If the fiber is T^2, then M^{k+2} is a closed $K(\pi, 1)$ and must be the standard T^{k+2} by [13; 7.2]. In the other cases we may enlarge the structure group to connected groups. We may by the same trick used in the lemma reduce the problem to $M^{k+2} = T^{k-4} \times X$, where $(T^2, X) = (T^2, T^2 \times_\Delta S)$. These bundles over T^2 are trivial if $\pi_1(T^2 \times_\Delta S)$ is to be abelian. In the non-injective case, the Reduction Theorem places us in the setting of the preceding lemma. Aside from the special cases avoided above, this theorem tells us that if $\pi_1(M^{k+2})$ is non-abelian then the action (T^k, M^{k+2}) must be locally injective. Moreover, all the orientable even dimensional M^{k+2} admit complex structures except possibly the special reductions.

To what extent (T^k, M^{k+2}) is a topological product when the action is locally injective seems to be a very interesting question and can be reduced to the case $k = 2$. In [16] we treat this problem in the homologically injective case and often find that the same M^4 may be written as a product $M_1^3 \times S^1$ and also as $M_2^3 \times S^1$ but $\pi_1(M_1^3) \neq \pi_1(M_2^3)$.

To close this section we point out that when M^{k+2} is a Seifert fiber space with generic fiber T^k and M^{k+2} is a closed $(k+2)$-dimensional manifold, then the principal T^k-bundle with operators N over the simply connected space W is a bundle over R^2 or S^2. The action of N on R^2 or S^2 is topologically linear. Let us assume that N is infinite and not the finite number of crystallographic possibilities. That is, N has a normal Fuchsian group of finite index and M is a $K(\pi, 1)$. Let $\phi: N \to \mathrm{Aut}(Z^k)$. Then using a suitable generalization of the central case, [13; 8.6], one can show that $\pi_1(M^{k+2})$ completely determines the diffeomorphism type of M^{k+2}. This general non-central case has already been obtained independently by H. Zieschang by a rather different approach in the announcement, [35]. In real dimension 4 this topologically classifies an interesting subclass of Kodaira's elliptic surfaces. The results appear to hold in the crystallographic cases as well.

13. C^*-Fiberings

Several times in the text we have constructed holomorphic Seifert fiberings from C^*-actions. It should be apparent that we could have used any abelian Lie group, G, (real or complex) in our constructions of Seifert fibre spaces without any but the obvious changes.

When the homomorphism $\phi: N \to \mathrm{Aut}\, G$ is trivial our Seifert fibre spaces are realized as actions:

$$(G, G \times W/N) \xrightarrow{/G} W/N \quad .$$

These actions are proper. An action (G,X) is proper if

(i) All stability groups are finite,

(ii) Each $x \in X$, has a neighborhood V so that

$$\{g \in G \mid gV \cap V \neq \emptyset\}$$

has compact closure in G.

If (W,N) is holomorphic, the map, induced by the action, $G \times X \to X$ is holomorphic. We are also led to the concepts of locally injective, injective, and homologically injective actions just as in the toral case.

Conversely, given a locally injective proper action, we may reconstruct the action as a "principal" Seifert fibre space because we have local slices. In fact condition (ii) is equivalent to the existence of local slices in the smooth and continuous cases by a theorem [31 ; 2.32] of R. Palais. For holomorphic actions, the existence of holomorphic slices has been proved by H. Holmann in [20]. Both conditions (i) and (ii) are required in Holmann's theorem.

Of course, the study of such actions has had a long history in complex manifolds and algebraic geometry. Recently P. Orlik and P. Wagreich, capitalizing on the work of F. Hirzebruch, E. Brieskorn, J. Milnor, R. Von Randow and others, have analyzed the topological nature of the singularities of special types of algebraic varieties by studying the C^*-action supported by these varieties, [30]. Fairly complete results were obtained by them for affine surfaces in C^n defined by weighted homogeneous polynomials. These varieties admit a proper C^*-action away from the singularities. By a construction they can reduce the study of singularities to isolated singularities. The proper holomorphic C^*-action on the non-singular surface, S, can then be reduced to studying the smooth S^1-action on a closed oriented 3-manifold. This closed 3-manifold, M^3 can be chosen to be the intersection with the variety of small sphere in C^n about the singularity. (Such actions were studied in the last section.) If $\pi_1(M^3)$ is not finite, and $M^3 \neq S^2 \times S^1$, then this holomorphic C^*-action

arises from an element in $H^1(N; \mathcal{C}^*)$. Its coboundary in $H^2(N; Z)$ is a Bieberbach class, where $N = \pi_1(S)/\mathrm{im}(f_x^x)$ acts holomorphically and properly discontinuously on the unit disk or the complex line and has compact quotient. (It is clear that S is diffeomorphic to $R^1 \times M^3$ since any continuous proper action of a non-compact abelian Lie group has the vector subgroup acting freely (and hence as a product).) However, not every possibility that arises holomorphically has a topological realization as an isolated algebraic singularity. In particular, we mention that according to Orlik and Wagreich $b + \sum_{i=1}^{i=n} (\beta_i/\alpha_i)$ must be negative. We may make the variety S compact by moding out a free holomorphic action of Z (Z a closed subgroup of R^1 in C^*). Therefore by (12.13) the non-singular closed surface, S/Z, never admits a Kähler structure. Since the variety with singularity is topologically the cone over M^3, this is equivalent to saying that the induced S^1-action on M^3 can never be homologically injective.

The procedure of Orlik and Wagreich is to use the equation defining the variety, to determine topologically the invariants of the C^*-action (that is, the S^1-action on M^3) and then the C^*-action provides a canonical equivariant resolution of the singularity. It is probably hopeless to algebraically classify these singularities in terms of the equations, but the exact sequence

$$0 \longrightarrow Z \xrightarrow{\epsilon} C \xrightarrow{e} C^* \longrightarrow 10$$

can be used to help classify the singularities analytically. From (5.1) and (6.1) we obtain the exact sequence:

$$0 \longrightarrow \mathrm{Pic}(W/N) \longrightarrow H^1(N; C^*) \longrightarrow H^2(N; Z) \longrightarrow 0 .$$

In Pic(W/N) one finds the deformations. Let us fix the complex Lie group C^*, and consider two holomorphic actions (C^*, S_1) and (C^*, S_2) which can be represented by b_1 and b_2 in $H^1(N; \mathcal{C}^*)$. We assume that the classes are Bieberbach classes in $H^2(N; Z)$ and that (W, N) is the action of a fixed Fuchsian group on the unit disk. Then the holomorphic analogue of [13, 8.6] states that (C^*, S_1) and (C^*, S_2) are equivariantly holomorphically equivalent if one can find an analytic homeomorphism $\mathcal{I}: W \longrightarrow W$ and an automorphism $\circledast: N \longrightarrow N$ so that $\mathcal{I}(w\alpha) = \mathcal{I}(w)\circledast(\alpha)$, for all $w \in W$, $\alpha \in N$ and $\circledast^*(b_2) = b_1$. A complete classification (of moduli) must also take into account the possibly different complex Lie groups C^*, and the possible complex structures on W/N. We do not pursue this here. Orlik and Wagreich are investigating these questions from a more algebraic point of view. We have mentioned our approach to these problems since we believe a combination of both methods may yield the most comprehensive results.

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HOLOMORPHICALLY INJECTIVE COMPLEX TORAL ACTIONS

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1. INTRODUCTION

This note is intended as a sequel to Holomorphic Seifert Fibering [7] in which the main emphasis is placed on studying the structure of what we call holomorphically injective complex torus actions. By a complex torus action (T,M) we will in general mean an effective left action of a complex torus T on a connected complex space M such that the natural map $T \times M \to M$ is holomorphic. The action (T,M) is holomorphically injective if for each $x \in M$ the orbit map $f_x : T \to M$ defined by $f_x(t) = tx$ induces a surjection $f_x^* : h^{1,0}(M) \to h^{1,0}(T)$ of the space of closed holomorphic 1-forms on M (modulo exact 1-forms if M is not compact) to the space of holomorphic 1-forms on T. In order to formulate this condition it is necessary to define $h^{1,0}(M)$, and we will therefore always assume that M is the quotient of a properly discontinuous holomorphic action on a simply connected complex manifold. In the Appendix, we define the space of closed holomorphic 1-forms and develop a theory of the albanese variety $A(M)$ of a space M with a quotient complex structure.

Before stating our results for complex torus actions, let us make some remarks about such actions on compact complex manifolds. First of all, a classical result of Bochner and Montgomery states that the group $\mathrm{Aut}(M)$ of all holomorphic transformations of M is a complex

Lie group acting holomorphically on M. Thus a complex torus action (T,M) occurs exactly when $\mathrm{Aut}(M)$ contains a compact connected complex subgroup T. Secondly, all isotropy groups T_x, $x \in M$, of (T,M) are finite (even if M is not compact). For the action induces a faithful complex analytic representation of the identity component of T_x into $GL(n;C)$ which must, by the maximum principle, be constant. In particular, a result of Holmann [Satz 20;8] on complex torus actions with finite isotropy applies, and we may assert that the orbit space M/T is a complex space under the quotient structure induced from M.

The assumption that (T,M) is holomorphically injective allows us to relate the theory of complex torus actions to properly discontinuous actions as in [6]. Let us consider an example, due to Matsushima [9], which illustrates the main features of this reduction. Suppose M is a Hodge manifold whose first integral Chern class is of finite order; i.e., M is special. The albanese variety $A(M)$ of M is an abelian variety of dimension $1/2b_1(M)$ which, in this case, is finitely covered by the jacoby homomorphism $J':\mathrm{Auto}(M) \to A(M)$ defined on the identity component $\mathrm{Auto}(M)$ of $\mathrm{Aut}(M)$. Therefore $\mathrm{Auto}(M)$ is a complex torus and $(\mathrm{Auto}(M),M)$ is a complex torus action. We shall prove in section 4 that indeed it is holomorphically injective. Let Δ be the kernel of J', and let $F = J^{-1}(e)$, where $J:M \to A(M)$ is the jacoby map. It turns out that F is Hodge and that Δ operates on F. Then $\mathrm{Aut}(M) \times F$ is a finite covering of M with covering group Δ, and M is obtained from $\mathrm{Aut}(M) \times F$ by identifying $(t\delta,f)$ and $(t,\delta f)$. Therefore, every special Hodge manifold M is a holomorphic fibre bundle over $A(M)$ with fibre F and finite abelian structure group. It can be shown also that F is connected, special, and Hodge.

Let us now describe a classification for holomorphically injective actions on connected compact complex manifolds. Suppose first that

$M = V/K$ is a compact complex space with the quotient complex structure induced by a properly discontinuous holomorphic action (V,K) on a simply connected complex manifold V. Then every covering of M is of the same form. Associated to the group $N = \pi_1(M,x)/\mathrm{Im} f_{x*}$, we have a covering space B of M with a complex torus covering action (T,B) of (T,M). We can show that B has an albanese $A(B)$ and that $A(B)$ is isomorphic with $A(M)$ by the natural map. As a consequence, we obtain the following description. Suppose (T,M) is holomorphically injective. Then there exist: i) a simply connected complex space W and a properly discontinuous holomorphic action (W,N) such that W/N is compact; and ii) a homomorphism $m:N \to T$ which defines a principal action $(T \times W, N)$ by the formula $(t,w)\alpha = (tm(\alpha), w\alpha)$ such that (T,M) is equivalent to the natural action $(T, T \times_N W)$ obtained by letting T act on the first factor. This leads to a characterization in the nonsingular case. Namely, the holomorphically injective actions (T,M) in which M is a connected compact complex manifold are precisely the actions $(T, T \times_N W)$ described above, where W is a simply connected complex manifold.

Holomorphic injectivity is the complex analytic analogue of topological injectivity, the condition that each orbit map f_x induce a monomorphism of $H_1(T;Z)$ into $H_1(M;Z)$. Thus the topological structure of a holomorphically injective action (T,M) is already determined, since holomorphic injectivity implies homological injectivity. In fact, in [6] it is shown that M is a topological fibre bundle over T with finite abelian structure group. But this fibring is not in general complex analytic, so there remains the question of when M fibres holomorphically. This depends on the homomorphism $m:N \to T$ defining $T \times_N W$. Indeed, let G be the complex closure of the image of m in T, that is, the intersection of all closed complex subgroups of T containing the image of m. Then M is a holomorphic fibre bundle over T/G with connected fibre F and infinite struc-

ture group. In general, the structure group G is disconnected. Moreover, in the case that T is an abelian variety, there is an exact sequence

$$0 \to \mathfrak{F} \to A(F) \to A(M) \to T/G \to 0,$$

where $\mathfrak{F}$ is a finite abelian group.

A fundamental class of holomorphically injective actions is given by the complex torus actions (T,M) in which M is compact Kaehler. Since we may average the Kaehler metric on M so that T acts as a group of Kaehler isometries, we shall call such actions Kaehler actions. They are characterized by the fact that $M = T \times_N W$ where (W,N) is a properly discontinuous group of Kaehler transformations on W. If M is Hodge, a complex torus T acting holomorphically on M is necessarily an abelian variety, and, in that case, we shall call (T,M) an abelian action. We will prove the following fact about abelian actions; namely, M is a holomorphic fibre bundle over T/Δ with connected fibre and structure group Δ, where Δ is a finite subgroup of T. This fact is not true, in general, for Kaehler actions. In other words, for abelian actions the defining cocycle $m \in \mathrm{Hom}(N,T)$ is of finite order, although for Kaehler actions it need not be.

The paper is organized as follows. In section 2 we study the structure of holomorphically injective actions (T,M) where M is a compact connected complex space with a quotient structure and deduce our characterization for nonsingular M. In section 3, we derive a condition that M fibre holomorphically over a quotient T/G of T and mention a consequence of that fibering. In section 4, we derive a differential geometric characterization of holomorphically injective actions and use that to show that Kaehler actions, and hence abelian actions, are holomorphically injective. We also characterize abelian actions. Finally, we include an appendix in which is considered a space of closed holomorphic 1-forms and an albanese variety for certain quotient complex spaces. The Appendix will be used in section 2.

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2. HOLOMORPHICALLY INJECTIVE ACTIONS

Let us begin by reviewing a canonical construction of Holomorphic Seifert Fibrations, given here in a simplified form, which associates to every properly discontinuous holomorphic action (W,N) on a complex space W and to every 1-cocycle in $Z^1(N;\text{ map }(W,T))$ a complex analytic space $M = T \times_N W$ and a complex torus action (T,M) with finite isotropy. Recall that a cocycle $m \in Z^1(N;\text{map }(W,T))$ can be identified with a holomorphic map $m: W \times N \to T$ satisfying the cocycle condition $m(w,\alpha\beta) = m(w,\alpha)m(w\alpha,\beta)$. To each cocycle, we associate an action $(T \times W,N)$ by setting

$$(t,w)\alpha = (tm(w,\alpha),w\alpha)$$

for all $t \in T$, $w \in W$, and $\alpha \in N$. It can be shown that the elements of $Z^1(N;\text{map }(W,T))$ are in a one to one correspondence with the properly discontinuous holomorphic actions $(T \times W,N)$ compatible with (W,N) . This may be seen with the aid of a lemma (stated without proof).

<u>Lemma 1</u>. <u>Let W and Y be locally compact Hausdorff spaces with Y compact. Suppose one is given a topological action (W,N) and a cocycle $m \in Z^1(N;\mathfrak{F}(W,Y))$, where $\mathfrak{F}(W,Y)$ denotes the space of continuous maps of W into Y . Then, in order that the associated topological action $(Y \times W,N)$ be properly discontinuous, it is necessary and sufficient that (W,N) be properly discontinuous.</u>

Given a cocycle m and hence an action $(T \times W,N)$ we can form the space $T \times_N W$ by factoring the action of N . By a result of Serre, $T \times_N W$ is a complex space such that: i) the natural map $\nu: T \times W \to T \times_N W$ is holomorphic; and ii) a map f defined on

$T \times_N W$ is holomorphic if and only if $f\nu$ is holomorphic on $T \times W$. In other words, $T \times_N W$ has the quotient complex structure. Next, observe that there is a natural action of T on $T \times_N W$ defined by letting T act on the first factor; i.e., $t(s \times w) = (ts) \times w$. It can be seen readily that $(T, T \times_N W)$ is a complex torus action with respect to the quotient complex structure on $T \times_N W$.

Let N_w denote the isotropy group of (W,N) at w. Then the isotropy group at $t \times w$ of the T action is $m(w,N_w)$, and hence finite. For each $w \in W$ there is a homomorphism $\chi_w : N_w \to T$ defined by $\chi_w(\alpha) = m(w,\alpha)$. It is easy to see that $(T \times W, N)$ is principal if and only if each χ_w is an injection. In that case, the quotient space $(T \times_N W)/T$ is isomorphic to W/N.

From now on we shall restrict our attention to properly discontinuous holomorphic actions (W,N) where W is a simply connected complex space and W/N is compact (always understood to have the quotient complex structure). For any complex space Y the set $\mathrm{map}(W,Y)$ of all holomorphic maps of W into Y is acted on by N as follows: for $\alpha \in N$ and $f \in \mathrm{map}\,(W,Y)$, set $(\alpha_\# f)(w) = f(w\alpha)$. Thus, if Y is either a complex torus T or C^k, then $\mathrm{map}\,(W,Y)$ becomes a $Z(N)$-module. Recall that a complex torus is defined by an exact sequence

$$0 \to Z^{2k} \xrightarrow{\epsilon} C^k \to T \to 1 ,$$

where ϵ is a homomorphism whose image is a lattice in C^k. Since W is simply connected, one therefore obtains an exact sequence of $Z(N)$-modules

$$0 \to Z^{2k} \to \mathrm{map}\,(W,C^k) \to \mathrm{map}\,(W,T) \to 1 ,$$

where we view Z^{2k} as a trivial $Z(N)$-module. Hence there is a cohomology exact sequence

$$\to H^i(N;\mathrm{map}\,(W,C^k)) \to H^i(N;\mathrm{map}\,(W,T)) \to H^{i+1}(N;Z^{2k}) \to$$

We turn now to the study of holomorphically injective actions (T,M) where M is the quotient space of an action (V,K) on a simply connected complex manifold V. M is always assumed to be compact. Let $f_x : T \to M$ denote the orbit map $f_x(t) = tx$. We wish to define an induced homomorphism $f_x^* : h^{1,0}(M) \to h^{1,0}(T)$. For a definition of a closed holomorphic 1-form see the Appendix. Let K_0 denote the normal subgroup of K generated by all elements of K having fixed points. Then V/K_0 is the universal covering space of M. Since (V,K) is properly discontinuous, all elements of K_0 have finite order. Therefore, if (g,φ) denotes a closed holomorphic 1-form on M, we see that g induces a holomorphic map $\tilde{g} : V/K_0 \to C$ and a homomorphism $\tilde{\varphi} : K/K_0 \to C$. For if $\alpha \in K_0$, $\varphi(\alpha) = 0$, and thus $g(v\alpha) = g(v) + \varphi(\alpha) = g(v)$ for all $v \in V$. Hence the unique lift $F_x : C^k \to V/K_0$ of f_x defines $(\tilde{g}F_x, \tilde{\varphi}f_{x*})$ as an element of $h^{1,0}(T)$, where f_{x*} denotes the induced homomorphism of $\pi_1(T,e)$ into $\pi_1(M,x) = K/K_0$.

Lemma 2. *f_x^* is independent of $x \in M$.*

By Lemma 1 of the Appendix, there is an imbedding $re : h^{1,0}(M) \to H^1(M;R)$. The independence of f_x^* on x thus follows from the independence of f_x^* on x in the homomorphism $H^1(M;R) \to H^1(T;R)$.

Lemma 3. *If (T,M) is holomorphically injective, then for each $x \in M$, f_x induces an injection of $H_1(T;Z)$ into $H_1(M;Z)$, i.e., (T,M) is homologically injective.*

The proof is straightforward. Since $\pi_1(T,e)$ is commutative, f_x induces an injection of $\pi_1(T,e)$ into $\pi_1(M,x)$, thus every holomorphically injective action is topologically injective also. By Lemma 4.2 of [5], the image of $\pi_1(T,e)$ by f_{x*} lies in the center of $\pi_1(M,x)$, and hence is a normal subgroup. Let N denote the quotient group $\pi_1(M,x)/\mathrm{Im} f_{x*}$. Our first main result is the following.

Theorem 1. *Suppose (T,M) is a holomorphically injective complex*

torus action on a compact connected complex space M *which is the quotient* V/K *of a properly discontinuous holomorphic action* (V,K) *on a simply connected complex manifold* V. *Then there exists a properly discontinuous holomorphic action* (W,N) *on a simply connected complex space* W *with* W/N *compact, and a cocycle* $m \in Z^1(N;\text{map}\,(W,T))$, *for which each* $\chi_w : N_w \to T$ *is injective, such that* (T,M) *is equivalent to the complex torus action* $(T, T \times_N W)$.

We will prove in Theorem 2 that holomorphically injective actions come only from the subgroup $\text{Hom}\,(N,T) \subset Z^1(N;\text{map}\,(W,T))$.

Let B denote the covering space of M associated to the group N. By Theorem 4.3 of [5], the torus action (T,M) can be lifted to a (topological) torus action (T,B) commuting with the action (B,N). (From this it is immediate that, with respect to the complex structure on B induced from M, (T,B) is a complex torus action, provided M is nonsingular.) We wish to prove two assertions. The First Assertion is that B has a complex structure such that the action (T,B) is a complex torus action; the Second Assertion is that we can define a jacoby map $J:B \to T$ such that $J(tb) = tJ(b)$. From the second assertion, it will follow that (T,B) is equivalent to $(T, T \times (B/T))$. The existence of a complex structure on B/T is guaranteed by a result of H. Holmann [*Satz* 20; 8].

Recall that $\pi_1(M,x)$ is canonically isomorphic with K/K_0, where K_0 denotes the normal subgroup of K generated by all isotropy groups of (V,K), or equivalently the least normal subgroup of K containing all isotropy groups of (V,K). Thus we can define a surjection $u:K \to N$ such that $K_0 \subset H = \ker u$. Let us redefine B to be V/H with the canonical quotient structure. It can be checked that we have a properly discontinuous holomorphic action $(V/H, K/H)$ which we write (B,N), since K/H is isomorphic with N.

Lemma 4. (B,N) *is principal*.

This follows since N is a quotient of $\pi_1(M,x)$ and thus

$H \supset K_0$. For if $b\alpha = b$, for some $\alpha \in N$ and $b \in B$, then there exist a $v \in V$, $\beta \in K$, and $j \in H$ such that $u(\beta) = \alpha$ and $v\beta = vj$. It follows that $\beta j^{-1} \in K_0 \subset H$, so $\beta \in H$. Therefore $\alpha = e$.

We have now constructed a complex structure for B so that (B,N) is a holomorphic covering action of M and so that $M = B/N$. By Theorem 4.3 of [5] there is a torus action (T,B) commuting with the action of N ; i.e., $(tb)\alpha = t(b\alpha)$ for all $t \in T$, $b \in B$, and $\alpha \in N$. Hence (T,B) is a complex torus action.

Although B is not compact, we can still associate to B a complex torus $A(B)$ and a ''jacoby map'' $J:B \to A(B)$. It will turn out that there is an isomorphism $J':T \to A(B)$ such that $J(tb) = J'(t)J(b)$. We first consider the space of closed holomorphic 1-forms on B with respect to (V,H) , namely the set of all pairs (f,φ) , where:

i) $f:V \to C$ is holomorphic and $f(v_0) = 0$; and

ii) $\varphi:H \to C$ is a homomorphism such that, for all $v \in V$ and $\alpha \in H$, $f(v\alpha) = f(v) + \varphi(\alpha)$.

Recall that for a closed holomorphic 1-form (f,φ) on B we have defined an element $re(f,\varphi) \in H^1(B;R)$. Let us say that (f,φ) is <u>exact</u> if $re(f,\varphi) = 0$, and let $h^{1,0}(B)$ denote the space of closed holomorphic 1-forms on B modulo the exact holomorphic 1-forms. The covering map $p:B \to M$ induces a homomorphism $p^*:h^{1,0}(M) \to h^{1,0}(B)$ so that the following diagram commutes:

$$\begin{tikzcd} h^{1,0}(T) & h^{1,0}(B) \arrow[l, "f_b^*"'] \\ & h^{1,0}(M) \arrow[u] \arrow[ul, "f_x^*"] \end{tikzcd}$$

where $x = p(b)$. Thus f_b^* is surjective. Moreover, by cohomological considerations, $H^1(B;R)$ and $H^1(T;R)$ have the same dimension, and thus it follows that f_b^* is an isomorphism. On the level of holomology, $f_{b*}:H_1(T;Z) \to H_1(B;Z)$ is an isomorphism, so we obtain a

commutative diagram

$$\begin{array}{ccc} h^{1,0}(T)^* & \longrightarrow & h^{1,0}(B)^* \\ \uparrow & & \uparrow \\ H_1(T;Z) & \longrightarrow & H_1(B;Z) \end{array}$$

where the horizontal arrows are isomorphisms induced by f_b. The first vertical arrow imbeds $H_1(T;Z)$ as a closed lattice in $h^{1,0}(T)^*$, so $H_1(B;Z)$ is imbedded as a closed lattice in $h^{1,0}(B)^*$. Thus f_b induces an isomorphism

$$J' : h^{1,0}(T)^*/H_1(T;Z) = T \to h^{1,0}(B)^*/H_1(B;Z) = A(B) .$$

Note that J' is independent of $b \in B$. The jacoby map $J: B \to A(B)$ is defined as for the albanese. We need to check that $J(tb) = J'(t)J(b)$. One may note that if $J(b) = e$, then $J(tb) = J'(t)$ by appealing to the definition of J. Then, since B is connected, and since the set of isomorphisms of T into $A(B)$ is discrete, we see that $J(tb)J(b)^{-1}$ is independent of $b \in B$, completing the verification.

Let $W \subset B$ denote $J^{-1}(e)$. We wish to put a complex structure on W so that the map $\varphi: T \times W \to B$ defined by $\varphi(t,w) = tw$ is holomorphic. It is easy to show, using the properties of J and J', that φ is bijective. This establishes a natural identification between W and B/T. But since (T,B) is principal, B/T has a natural (quotient) complex structure which makes φ holomorphic, and W is nonsingular if and only if M is. Therefore, (T,B) is equivalent to the natural action $(T, T \times W)$, and since the actions of T and N on $T \times W$ commute, the action $(T \times W, N)$ is defined by a cocycle $m \in Z^1(N; \text{map }(W,T))$ and a properly discontinuous holomorphic action (W,N) (Lemma 1). Finally, observe that W is simply connected, from the definition of B, and that $W/N = M/T$ is compact.

Observe that as a corollary we can state that the isotropy groups of a holomorphically injective action (T,M) are always finite. Hence, by the Appendix, there is an albanese exact sequence

$$T \to A(M) \to A(M/T) \to 0 .$$

Theorem 2. *Suppose (T,M) is a holomorphically injective torus action as in Theorem 1. Then the defining cocycle m is cohomologous in $H^1(N;\text{map }(W,T))$ to an element of $\text{Hom }(N,T)$.*

Before proving Theorem 2 we need to discuss the universal covering space of $M = T \times_N W$. Recall that T is defined by an exact sequence

$$0 \to Z^{2k} \xrightarrow{\epsilon} C^k \xrightarrow{\nu} T \to 1 .$$

Since W is simply connected, the map $m:W \times N \to T$ can be lifted to a holomorphic map $u:W \times N \to C^k$ such that $u(w,1) = 0$, for all $w \in W$, and $\nu u = m$. By the cocycle condition, for each $w \in W$,

$$u(w,\alpha) - u(w,\alpha\beta) + u(w\alpha,\beta) \in \epsilon[Z^{2k}] ,$$

so, by the exactness, there exists for each $(\alpha,\beta) \in N \times N$ a unique element $c(\alpha,\beta) \in Z^{2k}$, independent of w, such that $\epsilon c(\alpha,\beta) = u(w,\alpha) - u(w,\alpha\beta) + u(w\alpha,\beta)$. c is the extension cocycle associated to $\delta([m]) \in H^2(N;Z^{2k})$. Consider the associated group extension

$$0 \to Z^{2k} \to \pi \to N \to 1 ,$$

where $\pi = Z^{2k} \times N$, with group operation $(p,\alpha)(q,\beta) = (p+q+c(\alpha,\beta),\alpha\beta)$. We define a principal action $(C^k \times W,\pi)$ by

$$(v,w)(p,\alpha) = (v+\epsilon(p) + u(w,\alpha),w\alpha) .$$

This defines $C^k \times W$ as the universal covering space of M as well as the covering action of $\pi_1(M) = \pi$ on $C^k \times W$.

Let us now prove Theorem 2. Recall from the Appendix that a

closed holomorphic 1-form on M is a pair (g,φ) consisting of a holomorphic map $g:C^k \times W \to C$ with $g(w_0) = 0$, for some $w_0 \in W$, and a homomorphism $\varphi:\pi \to C$ such that

$$g(v+\epsilon(p) + u(\alpha), w\alpha) = g(v,w) + \varphi(p,\alpha) .$$

A holomorphic 1-form on T may be identified with a complex linear map $f:C^k \to C$. To say that $f_x^*:h^{1,0}(M) \to h^{1,0}(T)$ is onto is to say that there exists a holomorphic map $G:C^k \times W \to C^k$ and a homomorphism $\psi:\pi \to C^k$ such that $G(0,w_0) = 0$, $G(v,w_0) = v$ and (G,ψ) is a closed C^k-valued holomorphic 1-form, where we take for convenience $x = 1 \times w_0$. We wish to prove that there exists an $h \in \mathrm{map}\,(W,T)$ such that

$$m(w,\alpha)h(w\alpha)h(w)^{-1} \in \mathrm{Hom}\,(N,T)$$

for every $w \in W$. Fix $w \in W$ and define $F:C^k \to C^k$ such that $F(v) = G(v,w) - G(0,w)$. Since $G(v+\epsilon(p),w_0) = v + \epsilon(p) = G(v,w_0) + \psi(p,1)$, we get that

$$\begin{aligned} F(v+\epsilon(p)) &= G(v+\epsilon(p),w) - G(0,w) \\ &= G(v,w) - G(0,w) + \epsilon(p) . \end{aligned}$$

Thus $F(v+\epsilon(p)) = F(v) + \epsilon(p)$ for all $v \in C^k$, so $F = I$, and $G(v,w) = v + G(0,w)$ for all $v \in C^k$ and $w \in W$. Hence let $H(w) = G(0,w) = G(v,w) - v$. We have

$$G(v+\epsilon(p) + u(w,\alpha), w\alpha) = G(v,w) + \psi(p,\alpha) ,$$

so

$$v + \epsilon(p) + u(w,\alpha) + H(w\alpha) = v + H(w) + \psi(p,\alpha) ,$$

and therefore

(1) $$u(w,\alpha) = H(w) - H(w\alpha) + \psi(0,\alpha) .$$

Now $\alpha \to \psi(0,\alpha)$ is not a homomorphism, but notice that $(0,\alpha)(0,\beta)(0,\alpha\beta)^{-1} = (c(\alpha,\beta),1)$, so that, since $\psi(p,1) = \epsilon(p)$,

$$\psi(0,\alpha) + \psi(0,\beta) - \psi(0,\alpha\beta) = \epsilon c(\alpha,\beta) .$$

Thus $\tilde{m}(\alpha) = \nu\psi(0,\alpha)$ is an element of Hom (N,T), and by (1)

$$\tilde{m}(\alpha) = m(w,\alpha)h(w\alpha)h(w)^{-1}$$

for all $w \in W$ and $\alpha \in N$. Hence m and $\tilde{m}$ are cohomologous in $Z^1(M;\text{map }(W,T))$.

We can therefore regard the cocycle m of Theorem 1 as an element of Hom (N,T) since cohomologous cocycles of $Z^1(N;\text{map }(W,T))$ define equivalent actions (T,M).

Unfortunately, we cannot state a converse to Theorem 1. We can, however, prove a converse for nonsingular manifolds. In fact, we have the following classification.

Theorem 3. A complex torus action (T,M) _on a compact connected complex manifold is holomorphically injective if and only if there exists a properly discontinuous holomorphic action_ (W,N) _on a simply connected complex manifold_ W _and an element_ m _of_ Hom (N,T) _such that the properly discontinuous action_ $(T \times W,N)$ _given by_ $(t,w)\alpha = (tm(\alpha),w\alpha)$ _is principal and_ (T,M) _is equivalent to_ $(T,T \times_N W)$.

We have proved the necessity in Theorems 1 and 2. The proof of the sufficiency is not hard. Moreover, for a holomorphically injective action (T,M), we are guaranteed by [6] that N admits a normal subgroup L of finite index for which N/L is abelian and (W,L) is principal. We can therefore write $M = T \times_N (W/L)$ where the action $(W/L,N)$ is induced by the quotient homomorphism $N \to N/L$.

Let us now give some examples of complex torus actions.

Example 1. The Hopf manifold. Consider the properly discontinuous principal holomorphic action $(C^m - 0,\Delta)$ generated by $\zeta \to 2\zeta$. The quotient $(C^m - 0)/\Delta$ is called the Hopf manifold. We have also a canonical C^* action on $C^m - 0$ which defines the Hopf bundle $(C^m - 0) \to CP^{m-1}$. These actions commute, hence induce a holomorphic

action $(C^*/\Delta,(C^m - 0)/\Delta)$. Note that C^*/Δ is the complex torus of dimension one whose periods are $2\pi i$ and $\log 2$. The principal action $(C^*/\Delta,(C^m - 0)/\Delta)$ explicitly describes the Hopf manifold as a principal torus bundle over CP^{m-1} . Since the universal covering space of the Hopf manifold is $C^m - 0$, it cannot arise from a class in $H^1(N;\text{map }(W,T))$.

Example 2. (Blanchard [1; 160-161]) Another way of describing the Hopf manifold is the following. Let T denote the complex 1-torus with periods $2\pi i$ and $\log 2$. Then there exists a well defined homomorphism $C^* \to T$ given by $\lambda \to [\log \lambda]$, and hence a principal action $(C^*, T \times (C^m - 0))$ whose quotient space is the Hopf manifold. More generally, let T_2 be the complex 2-torus with periods $(2\pi i,0)$, $(\log 2,1)$, $(0,i)$, $(0,g) \in C^2$, with $g \in R$ irrational. As above, we have a principal action $(C^*,T_2 \times (C^m - 0))$ defined by

$$\lambda(\langle z_1,z_2\rangle,\zeta) = (\langle \log \lambda + z_1,z_2\rangle,\lambda\zeta) ,$$

whose quotient is a holomorphic fibre bundle M over CP^{m-1} with fibre T_2 . The natural complex torus action (T_2,M) is not holomorphically injective since $b_1(M) = 3$. For $b_1(M) = 3$ implies there can be at most one linearly independent closed holomorphic 1-form on M . In fact, if we denote an arbitrary point of T_2 by $\langle z_1,z_2\rangle$ with z_1,z_2 the standard coordinates on C^2 , then dz_2 defines a closed holomorphic 1-form on M . Now T_2 contains a one dimensional complex torus T_1 corresponding to the periods $(0,i)$, $(0,g) \in C^2$, and obviously the action (T_1,M) is holomorphically injective since dz_2 is a holomorphic 1-form on T_1 . Since the action (T_1,M) is principal, the defining covering action of M of Theorem 2 is $(T_1 \times (C^m - 0),Z)$, where the action of Z is given by the expression

$$(t,\zeta)n = (t\varphi(-n),2^n\zeta) ,$$

in which $n \to \varphi(n)$ is the homomorphism of $\mathrm{Hom}\,(Z,T_1)$ induced by the inclusion $n \to (0,n) \in C^2$. We may verify this assertion by noting that the action of Z on the second factor is the universal covering action of the Hopf manifold. To verify that the action on the first factor is correct, note that the homomorphism $Z \to C^*$ given by $n \to 2^n$ induces a bijective holomorphic mapping

$$\mu : T_1 \times_Z (C^m - 0) \to T_2 \times_{C^*} (C^m - 0) ,$$

where

$$\mu(\langle 0,z_2\rangle \times_Z \zeta) = \langle 0,z_2\rangle \times_{C^*} \zeta .$$

μ is clearly surjective. To see that μ is injective, suppose $\zeta \times_{C^*} \langle 0,z_2\rangle = \zeta' \times_{C^*} \langle 0,z_2'\rangle$. Then $\lambda\zeta = \zeta'$, for some $\lambda \in C^*$, and $\langle \log \lambda, z_2 - z_2'\rangle = 1 \in T_2$. Since g is irrational, we can assume that $(\log \lambda, z_2 - z_2') = (n \log 2, n)$, for some integer n, hence $\lambda = 2^n$, and therefore μ is injective. Finally, we see that the action $(T_1 \times (C^m - 0),Z)$ is

$$\begin{aligned}(\langle 0,z_2\rangle,\zeta)n &= (2^n\langle 0,z_2\rangle,2^n\zeta)\\ &= (\langle n \log 2,z_2\rangle,2^n\zeta)\\ &= (\langle 0,z_2 - n\rangle,2^n\zeta)\\ &= (\langle 0,z_2\rangle\varphi(-n),2^n\zeta) .\end{aligned}$$

Note that the periods of $dz_2 \in h^{1,0}(M)$ are $1,i,g$ and hence the image Γ of $H_1(M;Z)$ in $h^{1,0}(M)^*$ cannot be closed, since g is irrational, and therefore the albanese $A(M) = h^{1,0}(M)^*/\overline{\Gamma}$ vanishes.

Example 3. Let G be a connected compact semi-simple Lie group with a left invariant complex structure, and let T be a maximal torus in G equipped with a complex structure such that T is a complex submanifold of G (see [10]). Then T acts holomorphically on G on the right. Since the universal covering group of G is compact, G has no closed holomorphic 1-forms. However, let $\mathcal{R}$ denote the

space of right invariant 1-forms on G. Each right invariant 1-form on G is holomorphic, and, for every $g \in G$, f_g^* maps $\mathcal{R}$ onto $h^{1,0}(T)$.

Let us now make some remarks about the closed holomorphic 1-forms on M, when we are given a holomorphically injective complex torus action (T,M). In general, for any $x \in M$, $h^{1,0}(M) \cong h^{1,0}(T) \oplus$ $\oplus \ker f_x^*$ and we wish to consider $\ker f_x^*$. Let us write $M = T \times_N W$ as in Theorem 1 where the action $(T \times W, N)$ is given by a cocycle $m \in \mathrm{Hom}\,(N,T)$. There is a well defined homomorphism $\pi^*: h^{1,0}(M/T) \to h^{1,0}(M)$ induced by $\pi: M \to M/T$ whose image is a subspace of $\ker f_x^*$. Recall that f_x^* is independent of $x \in M$, and hence $\ker f_x^*$ consists of all $(g,\varphi) \in h^{1,0}(M)$ such that $g: C^k \times W \to C$ satisfies $g(v,w) = h(w)$ for all $(v,w) \in C^k \times W$. To determine the image of π^*, we seek a condition for $h \in \mathrm{map}\,(W,C)$ defined by $h(w) = g(0,w)$ to define a holomorphic 1-form on $M/T =$ $= W/N$. Now for $\alpha \in N$,

$$\begin{aligned} h(w\alpha) &= g(\mu(\alpha), w\alpha) \\ &= g[(0,w)(0,\alpha)] \\ &= g(0,w) + \varphi(0,\alpha) \\ &= h(w) + \varphi(0,\alpha), \end{aligned}$$

where $\mu: N \to C^k$ is a lift of m such that $\mu(1) = 0$. Thus a necessary and sufficient condition that $(h,\varphi) \in \ker f_x^*$ determines a holomorphic 1-form on W/N is that $\alpha \to \varphi(0,\alpha) \in \mathrm{Hom}\,(N,C)$. Now

$$\varphi[(0,\alpha)(0,\beta)(0,\alpha\beta)^{-1}] = \varphi(c(\alpha,\beta),1),$$

where $c \in Z^2(N;Z^{2k})$ is the extension cocycle representing $\delta(m)$, so the condition is that φ annihilate the image of c. In particular, if $m \in \mathrm{Hom}\,(N,T)$ lifts to a homomorphism $\widetilde{m} \in \mathrm{Hom}\,(N,C^k)$, the extension cocycle c vanishes and π^* is surjective. (This is precise-

ly what occurs in Example 2, where $N = Z$ and $m(n) = \langle 0,-n\rangle$.) Note that π^* is always injective. We may therefore state the following theorem.

Theorem 4. *Suppose (T,M) is a complex torus action on a compact complex space M associated to a properly discontinuous holomorphic action $(T \times W,N)$ via $m \in \mathrm{Hom}(N,T)$, where (W,N) is a properly discontinuous holomorphic action on a simply connected complex manifold W. Then if $\delta(m) = 0 \in H^2(N;Z^{2k})$, it follows that*

$$h^{1,0}(M) \cong h^{1,0}(T) \oplus h^{1,0}(W/N) .$$

Note that if we require that M be nonsingular and $\delta(m) = 0$, then (T,M) turns out to be a principal action, since for each $w \in W$, the trivial extension of N_w by Z^{2k} has elements of finite order unless N_w is trivial.

2. FIBERING THEOREMS

In this section we shall further discuss the structure of holomorphically injective torus actions (T,M). Recall that by Theorem 4.3 of [6], M is a topological fibre bundle over T with finite abelian structure group. However, the fibering is in general not holomorphic, so we wish to consider a condition that is sufficient for M to holomorphically fibre over a complex torus. Our condition will, in fact, pertain to a wider class of complex torus actions. To be as general as possible, we only assume that M is the quotient of a properly discontinuous holomorphic action $(T \times W,N)$ arising from a properly discontinuous action (W,N) on a connected complex space W and an element m of $\mathrm{Hom}(N,T)$.

Theorem 5. *Let $G \subset T$ be the smallest closed complex subgroup of T containing $m(N)$. Then M is a holomorphic fibre bundle over T/G with fibre $G \times_N W$ and infinite abelian structure group. Moreover, the fibre is connected.*

We can define a principal holomorphic action of G on $T \times (G \times_N W)$ by

$$h(t, g \times w) = (ht, hg \times w) .$$

The quotient $T \times_G (G \times_N W)$ is a holomorphic fibre bundle over T/G with fibre $G \times_N W$. Define a holomorphic map $\tilde{u}: T \times (G \times_N W) \to T \times_N W$ by $\tilde{u}(t, g \times w) = t^{-1}g \times w$. Then for all $h \in G$, $\tilde{u}(ht, hg \times w) = \tilde{u}(t, g \times w)$, so $\tilde{u}$ induces a holomorphic mapping $u: T \times_G (G \times_N W) \to T \times_N W$. u is clearly surjective, so to prove the theorem we have only to show that u is injective. Suppose then that, for $t, \tilde{t} \in T$, $g, \tilde{g} \in G$, and $w, \tilde{w} \in W$, we have $t^{-1}g \times w = \tilde{t}^{-1}\tilde{g} \times \tilde{w}$ in $T \times_N W$. Then, for some $\alpha \in N$,

$$\begin{aligned} (t^{-1}g, w)\alpha &= (t^{-1}gm(\alpha), w\alpha) \\ &= (\tilde{t}^{-1}\tilde{g}, \tilde{w}) , \end{aligned}$$

so $\tilde{w} = w\alpha$ and $t^{-1}gm(\alpha) = \tilde{t}^{-1}\tilde{g}$. We wish to prove that $t \times (g \times w) = \tilde{t} \times (\tilde{g} \times \tilde{w})$. Since $m(\alpha) \in G$, $h = t\tilde{t}^{-1} \in G$, and

$$\begin{aligned} h(\tilde{t}, \tilde{g} \times \tilde{w}) &= (h\tilde{t}, h\tilde{g} \times \tilde{w}) \\ &= (t, t\tilde{t}^{-1}\tilde{g} \times \tilde{w}) \\ &= (t, gm(\alpha) \times w\alpha) \\ &= (t, g \times w) . \end{aligned}$$

Therefore, $t \times (g \times w) = \tilde{t} \times (\tilde{g} \times \tilde{w})$ in $T \times_G (G \times_N W)$, and u is injective.

To finish the proof, we must show that $G \times_N W$ is connected. Let $K \subset N$ be the kernel of the defining homomorphism $m \in \mathrm{Hom}(N, T)$, and let G_0 be the identity component of G. We wish to define a surjective holomorphic map

$$\nu: G_0 \times W/K \to G \times_N W .$$

To do so, set $\nu(g_0,[w]) = g_0 \times w$, where $[w]$ denotes the equivalence class of $w \in W$ in W/K. We assert that every component of G meets the image $m(N)$. In other words, $G = m(N)G_0$ since the latter group is a closed subgroup of T containing $m(N)$. Now let $g \times w \in G \times_N W$. Then $g = m(\alpha)g_0$, for some $\alpha \in N$ and $g_0 \in G_0$, and $\nu(g_0 \times [w\alpha^{-1}]) = g_0 \times w\alpha^{-1} = g_0 m(\alpha) \times w = g \times w$. Therefore, ν is surjective.

Let us now study actions of abelian varieties. When (T,M) is a holomorphically injective complex torus action of an abelian variety T on a compact complex space M, we can say more about the fibering

$$G \times_N W \to M \to T/G .$$

In fact, by the Theorem of Complete Reducibility of Poincaré [11], if G is a closed complex subgroup of T, there is a complex torus H in T such that $H \cap G$ is finite and $T = G_0 H$, where G_0 is the identity component of G. Let $M' = (G \times H) \times_N W = H \times (G \times_N W)$. Then M' finitely covers $M = T \times_N W$ via the covering map $\varphi((g,h) \times w) = gh^{-1} \times w$. Now φ^* defines an isomorphism between $h^{1,0}(M)$ and $h^{1,0}(M')$, so we may consider the sequence

$$\varphi_* H_1(M;Z) \subset H_1(M';Z) \to h^{1,0}(M')^*.$$

It follows that $A(M)$ and $A(M')$ are of the same dimension. We wish to apply this fact to the exact sequence

$$A(G \times_N W) \to A(M) \to A(T/G) \to 0$$

of [1], Proposition I.2.2. For since $A(M)$, $A(M')$, $A(H) \times A(G \times_N W)$, and $A(T/G) \times A(G \times_N W)$ all have the same dimension, we get the following fact.

<u>Theorem 6</u>. <u>Let</u> (T,M) <u>be a complex torus action on a compact complex space</u> M <u>which fibres holomorphically as</u> $G \times_N W \to M \to T/G$

where $G \subset T$ is a closed complex subgroup. Then for some finite subgroup $\mathfrak{F} \subset A(G \times_N W)$, we have an exact sequence

$$0 \to \mathfrak{F} \to A(G \times_N W) \to A(M) \to A(T/G) \to 0 .$$

4. KAEHLER ACTIONS

We will now prove that all Kaehler actions are holomorphically injective. Recall that a Kaehler action (T,M) consists of a complex torus T acting holomorphically on a compact Kaehler manifold as a group of Kaehler isometries. In general, if M is a hermitian manifold, then we may average the metric over T so that T acts as a group of isometries, and if the metric is Kaehler, the metric resulting from the averaging is still Kaehler. Let g denote any hermitian metric on M invariant under T. Then any 1-parameter complex subgroup of T generates a holomorphic vector field X of type $(1,0)$ such that $L_X g = 0$. The existence of such a vector field on M is equivalent to the existence of a (real) Killing field Y on M such that JY is Killing, where J denotes the complex structure tensor of M. It can be shown that if g is Kaehler, then a holomorphic vector field X satisfies $L_X g = 0$ if and only if $X = Y - iJY$ where Y, and hence JY, is parallel with respect to the metric.

Let h denote an arbitrary hermitian metric on M and let X be a holomorphic vector field on M such that $X = Y - iJY$, where Y and JY are both parallel in the torsionless riemannian connection ∇ associated to h. For a vector field V on M, let ξ_V denote the 1-form on M dual to V defined by the formula $\xi_V(W) = h(V,W)$.

Lemma 5. $\xi_X = \xi_Y - i\xi_{JY}$ is a closed 1-form of type $(0,1)$. Consequently, $\overline{\xi}_X$ is closed and holomorphic on M.

For if V,W are any vector fields on M, then

$$d\xi_Y(V,W) = V\xi_Y(W) - W\xi_Y(V) - \xi_Y([V,W])$$
$$= h(Y,\nabla_V W) - h(Y,\nabla_W V) - \xi_Y([V,W])$$
$$= 0 \ ,$$

since ∇ is torsion free. Similarly $d\xi_{JY} = 0$. Thus,

$$d\xi_X = d\xi_Y - id\xi_{JY} = 0 \ ,$$

and since $\xi_X = \xi_Y - i\xi_{JY}$ is of type $(0,1)$, $\overline{\xi}_X$ is a closed holomorphic 1-form on M.

Now suppose (T,M) is a complex torus action on a compact hermitian manifold with T acting as isometries. Suppose that every complex 1-parameter subgroup generates a holomorphic vector field X which is parallel in the above sense.

Lemma 6. *(T,M) is holomorphically injective.*

This follows since $\overline{\xi}_X(X) \neq 0$ and since $\overline{\xi}_X$ is a closed holomorphic 1-form on M.

Conversely, if (T,M) is a holomorphically injective action on a connected compact complex manifold M, then, using the fact that there is a covering action $(T \times W,N)$ defined by a properly discontinuous holomorphic action (W,N) and a homomorphism $m:N \to T$, we can construct a hermitian metric on M so that the one parameter complex subgroups of T generate parallel vector fields on M. Thus we can state a differential geometric characterization of holomorphically injective actions.

Theorem 7. *A complex torus action (T,M) on a compact complex manifold M is holomorphically injective if and only if there exists a hermitian metric on M such that, with respect to the associated torsionless riemannian connection, the holomorphic vector fields generated by complex one parameter subgroups of T are parallel vector fields.*

Corollary. *Every Kaehler action (T,M) is holomorphically in-*

jective. The Kaehler actions are exactly those holomorphically injective actions described in Theorem 3 in which (W,N) is a properly discontinuous action by holomorphic Kaehler transformations.

The first assertion is immediate. To verify the second assertion, write $M = T \times_N W$ and suppose that on $T \times W$ we introduce the pull back Kaehler structure coming from M. In this structure, N is a group of Kaehler isometries of $T \times W$, and since the Kaehler structure of M is invariant by T, the same is true for the Kaehler structure of $T \times W$ The assertion now follows from the fact that the actions of T and N on $T \times W$ commute.

Let us now classify all complex torus actions (T,M) for which M is a connected Hodge manifold. A necessary condition arises immediately, namely that T must be an abelian variety. For the jacoby homomorphism J' of T into $A(M)$ has finite kernel, and hence, T finitely covers its image $J'(T)$. But $A(M)$ is an abelian variety, since M is algebraic, hence so is $J'(T)$, and thus we see that T is also, by e.g. Kodaira's Theorem.

Theorem 8. Assume (T,M) is an abelian action. Then there exist a finite subgroup Δ of T, a connected Hodge manifold F, and a holomorphic action (F,Δ) such that (T,M) is equivalent to $(T, T \times_\Delta F)$. In particular, M is a holomorphic fibre bundle over T/Δ with finite abelian structure group Δ and fibre F. Conversely, if T is an abelian variety, Δ a finite subgroup of T and F a connected Hodge manifold on which Δ acts holomorphically, then $(T, T \times_\Delta F)$ is an abelian action.

The converse part of the theorem is clear. Recall that the jacoby map J and the jacoby homomorphism J' are related by the identity $J(tx) = J'(t)J(x)$, for all $t \in T$ and $x \in M$. By the holomorphic injectivity, the kernel of J' is finite. Let T' denote $J'(T)$. Then T' is an abelian subvariety of $A(M)$ finitely covered by T. By the Theorem of Complete Reducibility of Poincaré, there

exists an abelian variety $H \subset A(M)$ such that the map $T' \times H \to A(M)$ given by $(t,h) \to th^{-1}$ is a finite covering. Thus, if $\mathfrak{F}$ denotes $T' \cap H$, then $A(M)$ is a holomorphic fibre bundle over $T'/\mathfrak{F}$. Let $\varphi : A(M) \to T/\mathfrak{F}$ denote the projection, and let $h = \varphi J$. Then, for all $t \in T$ and $x \in M$,

$$\begin{aligned} h(tx) &= \varphi(J'(t)J(x)) \\ &= \varphi(J'(t))\varphi(J(x)) \\ &= \varphi(J'(t))h(x) . \end{aligned}$$

Let Δ be the kernel of $\varphi J'$. Δ is a finite subgroup of T. Now $e \in T'$ is a regular value of h, so $F = h^{-1}(e)$ is a compact complex submanifold of M, and hence is Hodge. It can be shown that F is connected. Δ operates holomorphically on F and thus we may define a principal action $(T \times F, \Delta)$ by $(t\delta, f) = (t, \delta^{-1}f)$. It is clear that $(T, T \times_\Delta Y)$ and (T, M) are equivalent.

There are examples of nonabelian Kaehler actions. One example, of course, is a nonabelian complex torus. In order to find a nontrivial example one can start with a properly discontinuous group N of Kaehler transformations of a simply connected Kaehler manifold W such that W/N is compact. Then, if $m : N \to T$ is a homomorphism such that $m(N)$ is infinite, then $(T, T \times_N W)$ is a nonabelian Kaehler action (provided T is given an invariant Kaehler structure), since $T \times_N W$ cannot fibre over a finite quotient of T.

APPENDIX

The albanese variety associated to a properly discontinuous action.

In this appendix we associate to every properly discontinuous holomorphic action (W,N) on a simply connected complex manifold W, with compact orbit space $V = W/N$, a complex torus $A(W,N)$, called the albanese variety of (W,N), with the universal property for holomorphic maps of V into complex tori, and we prove that $A(W,N)$ depends only on V. We also derive a general albanese exact sequence

$$A(T) \to A(M) \to A(M/T) \to 0 \tag{1}$$

for complex torus actions (T,M) where M is of the form W/N with (W,N) as above. When N acts freely, the above construction is due to Blanchard.

To every compact complex manifold V, we can associate a covering action (W,N), where W is the universal covering space of V and N is the fundamental group of V. W may be given an induced complex structure from V so that (W,N) is a properly discontinuous holomorphic action. N acts freely and $V = W/N$. However, for a nonprincipal action, W/N is an analytic space with singularities [4], hence in order to define the albanese of V we must first adopt a general definition of the space of closed holomorphic 1-forms on V. Let us fix a base point $w_0 \in W$.

Definition. For a complex vector space E, let $L(W,N;E)$ denote the complex vector space of all pairs (f,φ) where

1) $f:W \to E$ is a holomorphic mapping such that $f(w_0) = 0$; and

2) $\varphi:N \to E$ is a homomorphism such that $f(w\alpha) = f(w) + \varphi(\alpha)$, for all $w \in W$ and $\alpha \in N$.

A pair (f,φ) satisfying 1) and 2) is called a *closed holomorphic 1-form* on V, or, more properly (W,N). f is called an *E-valued additive holomorphic mapping*.

Let K denote the smallest normal subgroup of N containing all the isotropy groups N_w, $w \in W$, of (W,N). Then N/K is the fundamental group of V. Any homomorphism $\varphi:N \to C$ annihilates K since the isotropy groups N_w are all finite. Furthermore, as C is abelian, the commutator subgroup is annihilated. Thus φ induces a homomorphism

$$\tilde{\varphi}:N/K/_{[N/K,N/K]} = H_1(V;Z) \to C$$

Let L denote $L(W,N;C)$.

Lemma 1. The homomorphism of L into $\mathrm{Hom}\,(H_1(V;Z),R)$ given by $(f,\varphi) \to \mathrm{re}\,\tilde{\varphi}$ is injective.

Suppose $(f,\varphi) \neq 0$; i.e., f is not identically constant. If $\mathrm{re}\,\tilde{\varphi} = 0$, then $\mathrm{re}\,\varphi = 0$, and thus $\mathrm{re}\, f(w\alpha) = \mathrm{re}\, f(w)$. It follows that $\mathrm{re}\, f$ defines a continuous function on V. This function assumes an absolute maximum at some point $v \in V$, and hence $\mathrm{re}\, f$ assumes an absolute maximum at some point $w \in W$. But in local complex coordinates near w, $\mathrm{re}\, f$ is a harmonic function, and the existence of w contradicts the maximum principle, since f is not identically constant.

It follows that L is finite dimensional, in fact,

$$\dim_R L \leq b_1(V) .$$

Let L^* denote the complex dual of L. There is a canonical homomorphism $\psi:N \to L^*$ inducing a homomorphism $\psi_*:H_1(V;Z) \to L^*$. Simply define

$$\langle\psi(\alpha),(f,\varphi)\rangle = \varphi(\alpha)$$

for $\alpha \in N$. We assert that $\Gamma = \mathrm{im}\,\psi_*$ generates L^* over R. If not, there is a real functional $h:L^* \to R$ such that $h(\Gamma) = 0$. To h corresponds a closed holomorphic 1-form (f,φ) such that

$$h(\ell^*) = \mathrm{re}\langle\ell^*,(f,\varphi)\rangle .$$

But for $\alpha \in N$,

$$h(\psi(\alpha)) = \mathrm{re} \langle\psi(\alpha),(f,\varphi)\rangle$$
$$= \mathrm{re}\, \varphi(\alpha)$$
$$= 0 .$$

Thus $h(\Gamma) = 0$ if and only if $\mathrm{re}\, \varphi = 0$, and this can occur only if $h = 0$.

There is a natural map $p:W \to L^*$ given by $\langle p(w),(f,\varphi)\rangle = f(w) \in C$. We thus obtain a map $\hat{p}:W \to L^*/\Gamma$. But L^*/Γ is not in general a complex torus. To remedy this we must replace Γ by its complex closure, the subgroup $\overline{\Gamma}$ of L^* described as the intersection of all closed subgroups of L^* containing Γ whose identity components are complex subspaces of L^* . Define the albanese $A(W,N)$ of (W,N) to be the complex torus $L^*/\overline{\Gamma}$. Clearly, p induces a holomorphic map $J:V \to A(W,N)$, since, for all $w \in W$ and $\alpha \in N$,

$$p(w\alpha)(f,\varphi) = f(w\alpha)$$
$$= f(w) + \varphi(\alpha)$$
$$= f(w) + \langle\psi(\alpha),(f,\varphi)\rangle ,$$

and therefore $p(w\alpha) - p(w) = \psi(\alpha) \in \Gamma$. Thus $\hat{p}(w\alpha) = \hat{p}(w)$, and we may set $J([w]) = \hat{p}(w)$. J is called the Jacoby map.

We shall now prove the universal property of the albanese variety. Let $\nu:W \to V$ be the natural map, and set $v_0 = \nu(w_0)$. We wish to show that every holomorphic map h of V into a complex torus T such that $h(v_0) = e$ can be factored through $A(W,N)$. In fact, we shall prove that there exists a unique homomorphism $\sigma:A(W,N) \to T$ such that the diagram

$$\begin{tikzcd} V \arrow[rr, "h"] \arrow[dr, "J"'] & & T \\ & A(W,N) \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes. Represent T as E/G, where G is a closed lattice in E generated by the linearly independent vectors $\xi_1,\cdots,\xi_{2k}$. We then have an alternative description of T as follows: let $L(T)$ denote the space of closed holomorphic 1-forms on T. Then $L(T)$ is naturally isomorphic to E^*, since a pair (f,φ) is a holomorphic 1-form on T if and only if $f \in E^*$. The dual isomorphism between E and $L^*(T)$ carries G onto $H_1(T;Z) = \Sigma$. Since a complex torus is always Kaehlerian, Σ is closed in $L^*(T)$ and $T \cong L^*(T)/\Sigma$, and we have demonstrated an isomorphism between T and its albanese $A(T)$. Suppose, therefore, that $h:V \to A(T)$ is a holomorphic map with $h(v_0) = e$. We seek a complex homomorphism $\sigma:A(V) \to A(T)$ such that

$$\begin{tikzcd} V \arrow[rr, "h"] \arrow[dr, "J"'] & & A(T) \\ & A(V) \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes. We are going to define an $L^*(T)$-valued closed holomorphic 1-form (ω,Φ) on (W,N) such that the following diagram commutes:

(2)
$$\begin{tikzcd} W \arrow[r, "\omega"] \arrow[d, "\nu"'] & L^*(T) \arrow[d, "\pi"] \\ V \arrow[r, "h"] & A(T) \end{tikzcd}$$

We begin by defining $\omega(w_0) = 0$. Now let $w \in W$, and consider a path γ_w from w_0 to w. Then $h\nu\gamma_w$ is a path from $e \in A(T)$ to $h(\nu(w))$ and this path lifts uniquely to a path in $L^*(T)$ at 0. Call the endpoint of this path $\omega(w)$. Since W is simply connected, $\omega(w)$ depends only on the endpoints w_0 and w of γ_w and not the

path itself. We therefore have a well defined holomorphic map $\omega : W \to L^*(T)$ which makes 2) commute. We now wish to define a certain homomorphism $\Phi : N \to L^*(T)$ compatible with ω. For each $w \in W$, there is a well defined homomorphism $\Phi_w : N \to L^*(T)$ given by the composition

$$N \to \pi_1(V,\nu(w)) \to \pi_1(T,h(\nu(w))) \xrightarrow{\sim} H_1(T;Z) \to L^*(T)$$

and for another w, say $w' \in W$, we have a diagram

$$\begin{tikzcd}
 & \pi_1(V,v) \arrow[r, "h_*"] \arrow[d] & \pi_1(T,h(v)) \arrow[d] \arrow[dr] & \\
N \arrow[ur] \arrow[dr] & & & L^*(T) \\
 & \pi_1(V,v') \arrow[r, "h_*"] & \pi_1(T,h(v')) \arrow[ur] &
\end{tikzcd}$$

where $v = \nu(w)$ and $v' = \nu(w')$ and where the vertical arrows represent the usual isomorphisms. It is immediate that the middle square and second triangle commute. We will show that the first triangle commutes, and from this it follows that $\Phi_w = \Phi_{w'}$. First of all, we define the map $\tau_w : N \to \pi_1(V,v)$ as follows. Let $\gamma_{w,\alpha}$ be a path from w to $w\alpha$. Then $\nu\gamma_{w,\alpha}$ is a loop at v which uniquely defines $\tau_w(\alpha)$. Obviously, $\tau_w(\alpha\beta) = \tau_{w\alpha}(\beta)\tau_w(\alpha)$. Observe that although τ_w is not a homomorphism, $h_*\tau_w$ is, by the commutativity of $\pi_1(T,h(v))$. For any choice of path s in W from w to w', we can show that

$$(s\alpha)\gamma_{w,\alpha}s^{-1} \sim \gamma_{w',\alpha}$$

by a homotopy fixing w' and $w'\alpha$, so

$$(\nu s)(\nu\gamma_{w,\alpha})(\nu s)^{-1} \quad \text{and} \quad \nu\gamma_{w',\alpha}$$

define the same element of $\pi_1(V,\nu(w'))$, and hence $(\nu s)_*\tau_w = \tau_{w'}$. But $(\nu s)_*$ is the isomorphism of $\pi_1(V,v)$ onto $\pi_1(V,v')$ so $\Phi_w = \Phi_{w'}$ for all $w, w' \in W$. To finish the proof we will show that $\omega(w\alpha) - \omega(w) = \Phi_w(\alpha)$ for all $w \in W$ and $\alpha \in N$. Now

$$\begin{aligned}\Phi_w(\alpha) &= \mathrm{lift}_0(h\nu\gamma_{w\alpha}\,(h\nu\gamma_w)^{-1})(1) - \mathrm{lift}_0(h\nu\gamma_{w\alpha}\,(h\nu\gamma_w)^{-1})(0) \\ &= \mathrm{lift}_0\,(h\nu\gamma_{w\alpha})(1) - \mathrm{lift}_0(h\nu\gamma_w)(1) \\ &= \omega(w\alpha) - \omega(w)\ ,\end{aligned}$$

where lift_0 denotes the lift sending $e \in A(T)$ to the origin. We therefore obtain an additive holomorphic map (ω,Φ) with values in $L^*(T)$ covering $h:V \to A(T)$.

We can define a homomorphism $P:L^* \to L^*(T)$, where $L^* = L^*(W,N;C)$, by defining $P(k)f = k(f\omega,f\Phi)$ for $f \in L(T)$ and $k \in L^*$. For $\alpha \in N$ with image $\psi(\alpha) \in L^*$ under the natural map $N \to \Gamma \subset L^*$, we have

$$\begin{aligned}P(\psi(\alpha))f &= \psi(\alpha)(f\omega,f\Phi) \\ &= f\Phi(\alpha) \\ &= \Phi(\alpha)f\ .\end{aligned}$$

But by the definition of Φ , $\Phi(\alpha) \in \Sigma$ for all $\alpha \in N$, so $P(\Gamma) \subset \Sigma$. Since Σ is closed, $P(\overline{\Gamma}) \subset \Sigma$ as well. Hence P induces a complex homomorphism $\sigma:A(W,N) \to A(T)$. We have finally to prove that

$$\begin{tikzcd} V \arrow[r, "J"] \arrow[dr, "h"'] & A(W,N) \arrow[d, "\sigma"] \\ & A(T) \end{tikzcd}$$

commutes. But recall that we have a natural map $p:W \to L^*(W,N)$ inducing J such that

$$\begin{tikzcd} & L^* \arrow[dr, "P"] & \\ W \arrow[ur, "p"] \arrow[rr, "\omega"] & & L^*(\Gamma) \end{tikzcd}$$

commutes. Thus, since $J(v) = [p(w)]$, where $v = \nu(w)$,

$$\begin{aligned}\sigma J(v) &= \sigma[p(w)] \\ &= [Pp(w)]\end{aligned}$$

$$= [\omega(w)]$$

$$= h(v) .$$

It is clear that σ is unique.

It is not hard to show, using the universal factorization property of $A(W,N)$, that $A(W,N)$ depends only on W/N and hence we may adopt this as the albanese of V.

Suppose (T,M) is a complex torus action on a connected compact complex space M which can be written as the quotient of a simply connected complex manifold by a properly discontinuous group. For each $x \in M$ we have the orbit map $f_x(t) = tx$, and composing f_x with the jacoby map $J:M \to A(M)$ we obtain a holomorphic map $T \to A(M)$, which is therefore a homomorphism followed by a translation of $A(M)$. Consider now the holomorphic map $J':T \times M \to M$ defined by $(t,x) \to J(tx)J(x)^{-1}$. For each x, $J'(\ ,x) \in \mathrm{Hom}\,(T,A(M))$. But $\mathrm{Hom}\,(T,A(M))$ is discrete and since M is connected, J' is independent of x. The resulting homomorphism $J':T \to A(M)$ is called the <u>jacoby homomorphism</u> of the action (T,M). Obviously, $J(tx) = J'(t)J(x)$; that is, the jacoby map is equivariant with respect to the jacoby homomorphism.

We wish now to derive an albanese exact sequence. Suppose again that (T,M) is a complex torus action on a compact complex space $M = W/N$, where (W,N) is a properly discontinuous holomorphic action on a simply connected manifold W. We are interested in defining in general an albanese variety for the orbit space M/T without any restriction on M/T, which will coincide with the usual albanese of M/T when that space is defined in the above sense. To do this we need to make an assumption on the action (T,M) which will be satisfied when either M is nonsingular or the action is holomorphically injective (section 2). Assume that all the isotropy groups T_x of (T,M)

are finite. Then M/T possesses a complex structure, as a quotient of M, satisfying the following: i) the quotient map $p:M \to M/T$ is holomorphic, and ii) a map $f:M/T \to Y$ is holomorphic, where Y is a complex space, if and only if $fp:M \to Y$ is holomorphic. We define the albanese variety $A(M/T)$ in an obvious manner. Simply set $A(M/T) = A(M)/J'(T)$, where $J':T \to A(M)$ is the jacoby homomorphism. Thus we can write an exact sequence

$$T \xrightarrow{J'} A(M) \xrightarrow{p_*} A(M/T) \to 0 ,$$

where p_* denotes the canonical homomorphism. To complete the definition, we must define a jacoby map $j:M/T \to A(M/T)$ and prove that the pair $(A(M/T),j)$ has the universal property for holomorphic maps of M/T into complex tori. Given $v \in M/T, J(p^{-1}(v)) = J(x)J'(T)$, by the equivariance property of the jacoby map, where $x \in p^{-1}(v)$, so that $J(p^{-1}(v))$ projects to a point in $A(M/T)$. Hence there is a well defined map $j:M/T \to A(M/T)$ such that

$$\begin{array}{ccc} M & \xrightarrow{p} & M/T \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle j} \\ A(M) & \xrightarrow{p_*} & A(M/T) \end{array}$$

is a commutative diagram. It is obvious, from the characterization of holomorphic maps on M/T, that j is holomorphic. Using the universal factorization property for $A(M)$, we can verify that $(A(M/T),j)$ has the universal factorization property for maps of M/T into complex tori. It follows that $A(M/T)$ is always isomorphic to the albanese of M/T (by any definition) and that the sequence 1) is always exact (c.f. Blanchard, Proposition II.2.2), provided (T,M) has only finite isotropy.

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DERIVED ACTIONS

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1. Introduction

Let (S^1, X) be a homologically injective action. That is, the homomorphism

$$f_*^x : H_1(S^1, 1) \longrightarrow H_1(X, x)$$

induced by the evaluation map

$$f^x(t) = tx, \qquad t \in S^1$$

is a monomorphism. In [7] we showed that if $H_1(X; Z)$ was finitely generated, then the following are equivalent:

(a) (S^1, X) is homologically injective,

(b) there is an equivariant map $p: (S^1, X) \longrightarrow (S^1, S^1/Z_n)$,

(c) X fibers over the circle with fiber Y and finite cyclic structure group, Z_n

(d) (S^1, X) is equivariantly homeomorphic to $(S^1, S^1 \times_{Z_n} Y)$,

(e) the Bieberbach class $a \in H^2(N; Z)$ representing the injective action (S^1, X) is of finite order (dividing n).

The purpose of the first 2 sections is to study for a fixed homologically injective action (S^1, X) the different possible ways that this action may fiber over the circle. As it turns out, it is no harder to treat a general action of a compact Lie group. That is, we may assume (G, X) is an action of a compact Lie group G on X and that the action is equivariantly homeomorphic to $(G, G \times_H Y)$ where H is a closed subgroup of G.

Whereas we treat this general case in §2, we shall continue to explain our results in terms of circle actions. The question as to what is meant by different fiberings is a matter of definition. However, we will exhibit in §3 actions (S^1, X) where $(S^1, S^1 \times_{Z_n} Y_1)$ is equivariantly homeomorphic to $(S^1, S^1 \times_{Z_n} Y_2)$ in the strongest possible sense, but Y_1 and Y_2

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have different homotopy types. In fact, in §3, we give a complete determination of the possible fundamental groups arising from a given (Z_n, Y) with fixed point.

Definition: Two fiberings (Z_n, W_1, Φ_1) and (Z_n, W_2, Φ_2) are *strictly equivalent* if and only if there is an equivariant homeomorphism $\Theta(S^1, X) \longrightarrow (S^1, X)$ for which the diagrams:

$$\begin{tikzcd} X \arrow[rr, "\Theta"] \arrow[dr, "\nu"'] & & X \arrow[dl, "\nu"] \\ & X/S^1 & \end{tikzcd}$$

and

$$\begin{tikzcd} \left((S^1 \times W_1)/Z_n\right) \arrow[rr, "\Phi_2^{-1}\Theta\Phi_1"] \arrow[dr, "p_1"'] & & \left((S^1 \times W_2)/Z_n\right) \arrow[dl, "p_2"] \\ & S^1 & \end{tikzcd}$$

both commute, where $\Phi_i : (S^1, S^1 \times_{Z_n} W_i) \to (S^1, X)$ are equivariant homeomorphisms.

As a corollary to Theorem 2.9 we are able to classify the set of strict equivalence classes cohomologically.

Theorem: *Let X be an orientable manifold and (S^1, X) homologically injective. Then the set of strict equivalence classes of fiberings over the circle with structure group Z_n is in one-one correspondence with the elements of*

$$\frac{H^1(X/S^1; Z)}{n\,H^1(X/S^1; Z)},$$

provided the set of singular orbits is not of codimension 2. In the codimension 2 case the set is a quotient of the above group.

The proof uses Fox's theory of spreads to reduce the problem to the free case. What makes the free classification possible is the reduction to a bundle problem. This is achieved by finding representatives in the strict equivalence classes for each possible fibering of (S^1, X). This is expressed in two different ways. The first is just by the possible equivariant maps $p_f : (S^1, X) \longrightarrow (S^1, S^1/Z_n)$ and the second by a construction of *derived actions* (Z_n, Y_f), one for each map

$$f : X/S^1 = Y/Z_n \longrightarrow S^1/Z_n .$$

While the spaces (and actions) Y_f derived from a given action (Z_n, Y) may be very different from Y, we find in §12.1 that:

If (Z_p, Y) is a group of orientation preserving diffeomorphisms on a closed oriented 2k-manifold, and if (Z_p, Y_f) is any derived action, then the oriented equivariant bordism classes are equal. That is, $[Z_p, Y] = [Z_p, Y_f] \in \mathcal{O}^{SO}_{2k}(Z_p)$, p an odd prime. This means invariants which only depend upon their bordism class do not change. Such an invariant is the Lefschetz-Atiyah-Bott trace $Tr(Z_p, Y)$ which is used in our computations.

Perhaps we should point out that in studying equivariant fiberings (S^1, X) over $(S^1, S^1/Z_n)$ we are demanding that the fiber has structure group Z_n. It is not difficult to fiber, say, $S^1 \times Y$ over S^1 by choosing different cyclic covering spaces of Y. That is, if (Z_n, Y') is a free action so that $Y'/Z_n = Y$, then $(S^1, S^1 \times_{Z_n} Y') = (S^1, S^1 \times Y)$ when the only torsion element of $H^2(Y; Z)$ is 0., (For example, Y is a closed oriented surface.) Thus, X is fibered over S^1 with very distinct fibers Y' and Y but the structure groups are Z_n and e respectively. The differences between derived actions (Z_n, Y) and (Z_n, Y_f) are more subtle.

Sections 2 and 3 are not used again until §12 which is our attempt at generalizing our very conclusive results in dimension 3, of §9 and 10, to higher dimensions. Moreover, Sections 7 and 8 and Sections 9 and 10 depend, independently, only on Sections 4, 5 and 6. Thus the reader may look at several parts of this paper without proceeding consecutively.

In most of the remaining parts of the paper we continue to investigate the different ways one may equivariantly fiber circle actions. But now we fix the fiber Y although we allow the actions of Z_n to vary slightly. Basically, we are concerned with the following study. If we take $(S^1, X) = (S^1, S^1 \times_{Z_n} Y)$ then this action is represented by a Bieberbach class $a \in H^2(N; Z)$ of order n. Our problem is to describe the other Bieberbach classes ma, where $(m, n) = 1$, in terms of (S^1, X) and the actions of Z_n on Y. The fibered action ma may be constructed as follows. Take $Z_m \subseteq S^1$. In $(S^1, X) = (S^1, S^1 \times_{Z_n} Y)$ the action of Z_m is free. If we take the quotient, we induce the action:

$$(S^1, S^1 \times_{Z_n} Y) \xrightarrow{/Z_m} \left(S^1/Z_m, (S^1 \times_{Z_n} Y)/Z_m\right)$$

The action $\left(S^1/Z_m, (S^1 \times_{Z_n} Y)/Z_m\right)$ may be identified with $\left(S^1/Z_m, (S^1/Z_m \times_{Z_n^{(2)}} Y)\right)$. The action of $Z_n^{(2)}$ on Y is related <u>but different</u> from $Z_n = Z_n^{(1)}$ on Y. In fact, if $T_{(1)}$ and $T_{(2)}$ denote the respective generators of $Z_n^{(1)}$ and $Z_n^{(2)}$ corresponding to $\exp((2\pi i)/n)$, then $T_{(2)}(y) = T^q_{(1)}(y)$, for all y, where $qm \equiv 1$ modulo n. Let us put $X(1) = S^1 \times_{Z_n^{(1)}} Y = X$ and $X(q) = S^1 \times_{Z_n^{(2)}} Y$. Then

$$(S^1, S^1 \times_{Z_n^{(2)}} Y) = (S^1, X(q)) \approx (S^1/Z_m, S^1/Z_m \times_{Z_n^{(2)}} Y)$$

represents ma. By now taking r, with $(r,n) = 1$, we may iterate this process:

$$(S^1, X(q), Z_r \subset S^1) \xrightarrow{/Z_r} (S^1/Z_r, X(q)/Z_r) \quad .$$

In terms of the original $(S^1, X(1))$ this becomes:

$$(S^1/Z_m, S^1/Z_m \times_{Z_n^{(2)}} Y, Z_r) \xrightarrow{/Z_r} (S^1/Z_{mr}, S^1/Z_{mr} \times_{Z_n^{(3)}} Y)$$

where $(Z_n^{(3)}, Y)$ is generated by $T_{(3)}(y) = T^{qr^{-1}}(y)$. Thus, $X(q)/Z_r = X(qr^{-1})$, qr^{-1} is taken modulo n and represents (mr)a. In particular, if $mr \equiv 1(n)$, that is, if $r \equiv q \bmod n$ then

$$X(1) \xrightarrow{Z_{mq}} X(1)/Z_{mq} \approx X(q)/Z_q \approx X(1)$$

is a covering where the total space and base space are homeomorphic. Details are found in §4.

Now it is easily seen, §4, that $S^1 \times X(1)$ is homeomorphic to $S^1 \times X(q)$, for all q, $(q,n) = 1$. However, what is intriguing is that X(1) need not be homeomorphic to X(q), even when both are closed smooth manifolds. The major concern of the remaining sections is to develop techniques for determining when X(1) and X(q) are or are not homeomorphic when Y is a manifold.

The first type of such examples were, to our knowledge, given by L. Charlap in [2]. His examples arose from his classification of flat manifolds in terms of their holonomy groups. These examples may also be described by our method. We introduce what we call Charlap actions (Z_n, Y) where Z_n is a cyclic group of orientation preserving diffeomorphisms on a closed aspherical (that is, Y is a $K(\pi, 1)$) manifold. These yield a special type of (S^1, X). We assume that (Z_n, Y) has a fixed point, y_0, and that $H^1(Y/Z_n; Z) = 0$. Here the major fact (Theorem 6.4) is that X(q) and X(1) have the same homotopy type if and only if $(T^q)_* : \pi_1(Y, y_0) \longrightarrow \pi_1(Y, y_0)$ is conjugate to $(T^{\pm})_*$ in the outer automorphism group of $\pi \approx \pi_1(Y, y_0)$. Our efforts in §7 and 8 are bent toward solving this problem. In §7, Charlap's solution is given in terms of an invariant which lies in the group of ideal classes for the cyclotomic number field $Q(\lambda)$ obtained by adjoining the p-th roots of unity to Q. Here Y is a k-dimensional torus and $n = p$, a prime.

In Section 8, it is shown that the Index(T^q, Y) = $\pm$Index(T, Y) if X(q) has the same homotopy type as X(1). The indexes may be computed in terms of the Atiyah-Bott-Lefschetz trace formula. We exhibit by this method examples of closed 3-manifolds where X(1) and X(q) have different homotopy types. Other examples using the trace formula are found in §11 and 12.

Since the circle actions on 3-manifolds are all known, it is to be expected that in this case one could say much more. In fact, in Sections 9 and 10 a complete solution to the homeomorphism problem for X(1) and X(q) is given when X(1) is a 3-manifold. The form of the solution in these cases are completely straightforward and can be checked at a glance. Moreover, there are some by-products of this analysis such as an extension of the generalized Nielsen theorem.

Section 11 shows how one may generate new Charlap actions from old. This leads to Charlap actions on closed aspherical manifolds with $\pi_1(X(1))$ not isomorphic to $\pi_1(X(q))$ for all dimensions greater than 2.

In Section 13 we discuss a related but different and puzzling type of example. Section 5 is an algebraic formulation of §4 and we feel that it, especially Theorem 5.5, explains algebraically what is transpiring in Sections 10 and 12. In these sections we are able to formulate our solutions without the restrictive assumption of a Charlap action.

2. Equivariant fiberings over a homogeneous space and derived actions

We fix an action (G,X) of the compact Lie group G on the space X and we denote by $\nu: X \longrightarrow X/G$ the orbit map. Let $j: H \longrightarrow G$ be a fixed isomorphism of the compact group H onto a closed subgroup of G. We shall consider the homogeneous space G/H as a space of left cosets and that H acts from the left on G by $h \times g \longrightarrow gh^{-1}$. Thus we have the action $(G \times H, G)$ defined by

$$(g,h) \times g' \longrightarrow (gg')h^{-1} = g(g'h^{-1}) \ .$$

2.1. Definition: A fibering of (G,X) over G/H of type H is a pair consisting of

(i) an action (H,W) of the group H,

(ii) an equivariant homeomorphism

$$\Phi: (G, G \times_H W) \longrightarrow (G,X) \ .$$

Note the action $(H, G \times W)$ is given by

$$h \times (g,w) \longrightarrow (gh^{-1}, hw).$$

We denote the image of (g,w) under the orbit map by $((g,w))$. There is then a G-equivariant map $p_W: (G, G \times_H W) \longrightarrow (G, G/H)$ given by

$$p_W: ((g,w)) \longrightarrow gH \quad .$$

2.2. Definition: Two fiberings $(H, W_1; \Phi_1)$ and $(H, W_2; \Phi_2)$ of (G,X) over G/H of type H are strictly equivalent if and only if there is a G-equivariant homeomorphism $\Theta: (G,X) \longrightarrow (G,X)$ so that

$$\begin{tikzcd} X \arrow[rr, "\Theta"] \arrow[dr, "\nu"'] & & X \arrow[dl, "\nu"] \\ & X/G & \end{tikzcd}$$

and

$$\begin{tikzcd} (G, G \times_H W_1) \arrow[rr, "\Phi_2^{-1} \Theta \Phi_1"] \arrow[dr, "p_1"'] & & (G, G \times_H W_2) \arrow[dl, "p_2"] \\ & (G, G/H) & \end{tikzcd}$$

both commute.

The strict equivalence classification is a measurement of the different ways one can equivariantly fiber (G,X) over G/H. We wish to determine this set. We shall exhibit a construction for which, beginning with any one such fibering, we can find representatives for all the others.

Suppose $p:(G,X) \longrightarrow (G, G/H)$ is an equivariant map. The trivial coset $\{H\}$ in G/H is an H-slice in $(G, G/H)$. If we put $Y = p^{-1}\{H\} \subset X$, then Y is the pull-back of the H-slice $\{H\}$ and so is an H-slice in (G,X), [1;Ch. 8]. Hence (G,X) may be written as a fiber bundle with fiber Y, structure group H and base space G/H. In terms of equivariant maps we have:

$$\begin{tikzcd}
(G \times H, G) \arrow[d, "/H"] & (G \times H, G \times Y) \arrow[l] \arrow[r, "/G"] \arrow[d, "/H"] & (H, Y) \arrow[d, "/H"] \\
(G, G/H) & (G, G \times_H Y) = (G,X) \arrow[l, "p"'] \arrow[r, "/G", "\nu"'] & Y/H = X/G
\end{tikzcd}$$

2.3. Definition: Let us say that $(G,X;Y,p)$ fibers equivariantly over $(G,G/H)$ if $p:(G,X) \longrightarrow (G,G/H)$ is a G-equivariant map and $Y = p^{-1}\{H\}$.

2.4 Lemma: *For each fibering $(H,W;\phi)$ of (G,X) over G/H of type H, there is an equivariant fibering $(G, X; Y_f, p_f)$ over $(G,G/H)$ whose fibering $(H, Y_f; \phi_f)$ is strictly equivalent to $(H, W; \phi)$.*

Proof: The map $p_W:(G, G \times_H W)$ defined by $p_W((g,w)) = gH$ is an equivariant map. Now

$$p_W^{-1}\{H\} = W = \Big\{((g,w)) \,\Big|\, ((g,w)) = ((e,w'))\Big\} \quad .$$

The equivariant homeomorphism ϕ defines a subspace

$$Y_f = \phi(W)$$

of X. (The use of the subscript "f" associated to ϕ will have later another, but equivalent, meaning when H is normal.) We form $(G, G \times_H Y_f)$ and an equivariant homeomorphism ϕ_f onto (G,X) by

$$\phi_f\big((g, \phi((e,w)))\big) = \phi((g,w)) \quad .$$

For θ we may choose the identity. We have

$$\begin{tikzcd}
(G, G \times_H W) \arrow[r, "\Phi"] \arrow[dd, "p_W"] & (G,X) \arrow[rr, "1"] \arrow[dr, "\nu"] & & (G,X) \arrow[r, "\Phi_f^{-1}"] \arrow[dl, "\nu"'] & (G, G \times_H Y_f) \arrow[dd, "p_f"] \\
& & X/G & & \\
(G, G/H) \arrow[rrrr, leftrightarrow] & & & & (G, G/H)
\end{tikzcd}$$

which commutes. Here p_f is to be interpreted as

$$p_f\big((g, \Phi((e,w)))\big) = p_W \circ \Phi^{-1} \circ \Phi_f^{-1}\big((g, \Phi((e,w)))\big) = gH \ .$$

Note also that we have an equivariant homeomorphism θ_f and commuting maps

$$\begin{tikzcd}
(H,W) \arrow[rr, "\theta_f"] \arrow[dr, "\nu"'] & & (H, Y_f) \arrow[dl, "\nu"] \\
& W/H = Y/H &
\end{tikzcd} \ .$$

We define $\theta_f(w) = \Phi((e,w))$. Now $\theta_f(hw) = \Phi((e,hw)) = \Phi((eh,w)) = h \cdot \Phi((e,w)) = h \cdot \theta_f(w)$.

2.5. Definition: Let $p_f: (G,X) \longrightarrow (G, G/H)$ be an equivariant map. We shall call (H, Y_f) the derived action associated with the map p_f.

Two derived actions (H, Y_{f_0}) and (H, Y_{f_1}) are said to be strictly equivalent if and only if there is an H-equivariant homeomorphism $\theta: (H, Y_{f_0}) \longrightarrow (H, Y_{f_1})$ so that the diagram

$$\begin{tikzcd}
Y_{f_0} \arrow[rr, "\theta"] \arrow[dr] & & Y_{f_1} \arrow[dl] \\
& Y/H &
\end{tikzcd}$$

commutes.

2.6. Theorem: Two fiberings $(H, W_0; \Phi_0)$ and $(H, W_1; \Phi_1)$ of (G,X) over G/H of type H are strictly equivalent if and only if their associated derived actions are strictly equivalent.

Proof: Let us assume that there is an equivariant homeomorphism $\theta: (H, W_0) \longrightarrow (H, W_1)$. Define $\boldsymbol{\theta}: (G,X) \longrightarrow (G,X)$ by:

$$\Theta\left(\Phi_0((g,w))\right) = \Phi_1((g,\theta(w)))$$
$$\parallel \qquad\qquad\qquad \parallel$$
$$\Theta\left(\Phi_{f_0}\left((g,\ \Phi_0((e,w))\right)\right) = \Phi_{f_1}\left((g,\ \Phi_1((e,\ \theta(w)))\right) \quad .$$

The second line is given in terms of the associated derived actions. It is easy now to check that Θ yields a strict equivalence between the two fiberings.

Conversely, given Θ we may define $\theta:(H,\ Y_{f_0};\Phi_0) \longrightarrow (H,\ Y_{f_1};\Phi_1)$ by

$$\theta((e,y_0)) = \Phi_1^{-1} \circ \Theta \circ \Phi_0((e,y_0)) \quad .$$

Note that $h\theta((e,y_0)) = h\Phi_1^{-1}\Theta\Phi_0((e,y_0)) = \Phi_1^{-1}\Theta\Phi_0((e,hy_0)) = \theta((e,hy_0))$. Thus $h\Theta = \Theta h$ implies that $h\theta = \theta h$ and the associated derived actions are strictly equivalent.

The point of all the preceding has been to replace strict equivalence of the G-actions by strict equivalence of the H-actions on the fibers. Furthermore, we have found fixed representatives for the fibers and the H-actions as well as the fiberings, all in terms of H-slices in (G,X). We can now examine the free case. We shall also be able to reduce the non-free case to the free case when H is finite.

Let us fix a particular action $(G,X) = (G, G \times_H Y)$. All derived actions (H, Y_f), strictly equivalent to $(H, Y; 1)$ must, first of all, yield an equivariant G-homeomorphism covering the identity $Y/H = X/G \overset{1}{\longleftrightarrow} X/G$. This means that the strictly equivalent free actions must yield equivalent principal G-bundles over X/G. Since the principal G-bundle fibers over $(G,G/H)$ with fiber Y and structure group H, the structure group of (G,X) is reducible to the closed subgroup H. Let $a \in H^1(Y/H, \underset{\sim}{G})$, where $\underset{\sim}{G}$ is the sheaf of germs of continuous functions into G, represent the principal G-bundles (G,X) over X/G. Let $c \in H^1(Y/H; \underset{\sim}{H})$ be the principal H-bundle over Y/H representing (H,Y). Since a is represented by $(G, G \times_H Y)$ then $j^*(c) = a$ is a reduction of the structure group, $j^*: H^1(Y/H; \underset{\sim}{H}) \longrightarrow H^1(Y/H; \underset{\sim}{G})$.

2.7. <u>Lemma</u>: <u>The set of strict equivalence classes (relative to the choice a) are the set of all bundle reductions of</u> a (<u>to the subgroup</u> H), <u>that is, all elements</u> $b \in H^1(X/G; \underset{\sim}{H})$ <u>such that</u> $j^*(b) = a$.

In particular, if G is a torus T^k and H is a closed finite subgroup, then this reduces to the Bockstein exact sequence:

$$0 \longrightarrow H^1(Y/H; Z^k) \xrightarrow{\ i\ } H^1(Y/H; Z^k) \longrightarrow H^1(Y/H; H) \xrightarrow{\ \beta\ } H^2(Y/H; Z^k) \quad .$$

The choice a is an element of $H^2(Y/H; Z^k)$; the strict equivalence classes are the elements $b \in H^1(Y/H; H)$ which are carried into a by the bockstein β. The possible choices of b are obtained by taking all elements in $H^1(Y/H; Z^k)$ reducing them modulo H and adding these reductions to a fixed element c for which $j^*(c) = \beta(c) = a$. This set may be identified with the elements of

$$(2.8) \qquad \frac{H^1(Y/H; Z^k)}{i\left(H^1(Y/H; Z^k)\right)} .$$

To consider not necessarily free (H, Y) when H is finite we employ Fox's theory of spreads [10]. The orbit map $\mu: Y_f \longrightarrow Y/H$ of each derived action (H, Y_f) is a complete spread. Consider now the subset $Y^+ \subset Y$ consisting of all points in Y at which the isotropy subgroup of H is trivial. Then Y^+ is the largest invariant subspace on which H acts freely. We impose on (H, Y) the hypothesis

(*) The set $Y^+ \subset Y$ is open dense and locally connected in Y. (Y^+ is locally connected in Y if there is a basis of Y such that $V \cap Y^+$ is connected for every basic open set V. We also assume Y is locally connected.)

In this situation $\mu: Y \longrightarrow Y/H$ is the completion of the spread $\mu: Y^+ \longrightarrow Y^+/H$. In a similar vein $Y_f^+ \subset Y_f$ can also be defined and $\mu_f(Y_f^+) = Y^+/H \subset Y/H$. Furthermore, as (H, Y_f^+) is just the action derived from (H, Y^+) by restriction of (H, Y_f) derived from (H, Y), it is not difficult to see that $\mu_f: Y_f \longrightarrow Y/H$ is the completion of the spread $\mu_f: Y_f^+ \longrightarrow Y/H$. From Fox's uniqueness theorem on the completion of spreads it will follow

2.9. Theorem: Under the hypothesis (*) the derived actions (H, Y_{f_0}) and (H, Y_{f_1}) are strictly equivalent if and only if $(H, Y_{f_0}^+)$ and $(H, Y_{f_1}^+)$ are strictly equivalent.

The hypothesis (*) is known to be satisfied for a finite group of orientation preserving homeomorphisms H on an orientable cohomology manifold Y. For orientable (T^k, X) our action (H, Y) will be orientation preserving. (In general, (*) is rather commonly found in examples of actions of cyclic groups.)

We would like to obtain the theorem mentioned in the introduction. It seems convenient to introduce a more explicit description of derived actions. For this we assume that the subgroup H is normal but not yet necessarily finite. We pick a fixed fibering $(G, X) = (G, G \times_H Y)$. Consider a fibering $(H, W; \Phi)$ of (G, X) over G/H of type H. We have defined $\theta: (H, W) \longrightarrow (H, Y_f)$ already, and the equivariant map $p_f((g, y)) = p_W\left(\Phi^{-1}((g, y))\right)$. Using normality, there exists a function

$$f: X \longrightarrow G/H$$

so that we can find a unique solution to

$$p_f((g,y)) = p((g,y)) \cdot f((g,y))$$

for each $((g,y)) \in X = (G \times_H Y)$. However, $p_f((g,y)) = g \cdot p_f((e,y))$ and $p((g,y)) = g \cdot p((e,y))$; so consequently we have

$$g \cdot p_f((e,y)) = g \cdot p((e,y)) \cdot f((g,y)) \ .$$

On the other hand, we have

$$p_f((e,y)) = p((e,y)) \cdot f((e,y)) \ .$$

Thus

$$f((g,y)) = f((e,y)) \ .$$

That is, f is constant on orbits and really can be thought of as being defined on X/G. Hence for each $(H, W; \Phi)$ we may find a unique function $f: X/G \longrightarrow G/H$ as above.

Let us now define, for any function $f: X/G = Y/H \longrightarrow G/H$, a set

$$C_f = \left\{(g,y) \,\middle|\, gf(\nu(y)) \text{ is the trivial coset } \{H\}\right\} \ .$$

It is convenient to define an action $G \times H_{(1)} \times H_{(2)}$ on $G \times C_f$ by

$$g' \times (\bar{g}, (g,y)) \longrightarrow (g'\bar{g}, (g,y))$$

$$h_1 \times (\bar{g}, (g,y)) \longrightarrow (\bar{g}, (gh_1^{-1}, h_1 y))$$

$$h_2 \times (\bar{g}, (g,y)) \longrightarrow (\bar{g} h_2^{-1} (h_2 g, y)) \ .$$

These actions are well defined and commute.

We define now an equivariant map

$$\mathcal{Y}: (G \times H_{(1)}, G \times C_f) \longrightarrow (G \times H, G \times Y)$$

by

$$\mathcal{Y}(g', (g,y)) = (g'g, y) = g'(g,y) \ .$$

$\mathcal{Y}$ is precisely the orbit map:

$$(H_{(2)}, G \times C_f) \xrightarrow{H_{(2)}} (G \times C_f)/H_{(2)} \ .$$

Thus, $\mathscr{G}$ induces the G-equivariant $H_{(2)}$-orbit map

$$\mathscr{G}^*:\left(G \times H_{(2)}, (G \times C_f)/H_{(1)}\right) \xrightarrow{/H_{(2)}} (G, G \times_H Y) \ .$$

We may also consider the projections and the actions restricted to these projections. These yield:

$$\begin{array}{ccc}
(H_{(1)} \times H_{(2)}, G \times C_f) & \xrightarrow{/H_{(2)}} & (H_{(1)}, (G \times C_f)/H_2) \\
\big\downarrow \text{projection} & & \big\downarrow \text{projection} \\
(H_{(1)} \times H_{(2)}, C_f) & \xrightarrow{/H_{(2)}} & (H_{(1)}, C_f/H_{(2)}) = (H, Y) \\
\big\downarrow /H_{(1)} & & \big\downarrow /H_{(1)} \\
(H, Y_f) = (H_{(2)}, C_f/H_{(1)}) & \xrightarrow{/H_{(2)}} & Y/H
\end{array}$$

In summary we have

2.10. <u>Lemma</u>: <u>If</u> $(H, W; \Phi)$ <u>is a fibering of</u> (G,X) <u>over</u> G/H <u>of type</u> H <u>and</u> (H, Y_f) <u>the associated derived action, where</u> $f: X/G \longrightarrow G/H$, <u>then</u> $C_f/H_{(1)} = Y_f$, <u>and</u> $(H_{(2)}, C_f/H_{(1)}) = (H, Y_f)$.

<u>Proof</u>: $(g,y) \in C_f$ if and only if $gf(\nu(y)) = H$. Recall that $f(\nu(y)) = f((g,y))$ and $p_f((g,y)) = p((g,y))f((g,y)) = gf((g,y)) = gf(\nu(y))$. Since $p_f^{-1}\{H\} = Y_f$, we see that $C_f/H_{(1)} \subseteq Y_f$. On the other hand, if $((g,y)) \in Y_f$, then $p_f((g,y)) = \{H\}$, which says that $(g,y) \in C_f$.

Thus, given $(G,X) = (G, G \times_H Y)$ where H is normal in G, we may in a systematic way construct all derived actions (H, Y_f) by forming $C_f = \left\{(g,y) \mid gf(\nu(y)) = H\right\}$, where $f: Y/H = X/G \longrightarrow G/H$ is a continuous function. The subspace $Y_f = C_f/H_{(1)}$ of X is also $p_f^{-1}\{H\}$ where $p_f((g,y)) = p((g,y))f(\nu(y))$ is a G-equivariant map from (G,X) to $(G, G/H)$. If we let $\mathbf{E}$ be the set of all G-equivariant maps, and E_0 to be the set of all maps $X/G \longrightarrow G/H$, then the group E_0 acts transitively on $\mathbf{E}$ and with stabilizer the identity. For given $p:(G,X) \longrightarrow (G, G/H)$, the equivariant map from the representation $(G,X) = (G, G \times_H Y)$ by $p((g,y)) = gH$, we may form $p_f((g,y)) = p((g,y))\, f(\nu(y))$, for any function $f: Y/H \longrightarrow G/H$. Conversely, given any arbitrary $q((g,y))$, we saw that we may construct the unique $f: Y/H \longrightarrow G/H$ by $q((g,y)) = p((g,y))\, f(\nu(y)) = p_f(g,y)$. Clearly, the function $f: Y/H \longrightarrow G/H$, which is constant and equal to $\{H\}$, is the stabilizer of $\{p\}$.

Thus, there exists a one-one correspondence between E_0 and E, with E_0 acting transitively on the set E.

Let us interpret the free case 2.7 in terms of the set E_0. Let

$$\sigma : B_H \xrightarrow{G/H} B_G$$

be the principal G/H fibering from the universal classifying space for the closed normal subgroup H to the group G. For any $\gamma : Y/H \longrightarrow B_H$ and $f \in E_0$ we may form, using the right action of G/H on B_H, $\gamma_f = \gamma \cdot f$. If γ represents $c \in H^1(Y/H; \underset{\sim}{H})$ so that $\beta(c) = a$, that is $\sigma \circ \gamma$ represents a, then (H, Y_f) is represented by γ_f and $\sigma \circ \gamma_f$ is equal to $\sigma \circ \gamma$. Let us now assume H is finite and (H, Y) is not necessarily free but satisfies the hypothesis (*). Let γ^+ represent $c^+ \in H^1(Y^+/H; \underset{\sim}{H})$, then for $f^+ \in E_0^+$ we may form $(\gamma^+)_{f^+} : Y^+/H \longrightarrow B_H$. It is necessary, however, that $f^+ : Y^+/H \longrightarrow G/H$ be extendable to all of Y/H, for otherwise $p_f = p \cdot f$ can not be defined. Thus the group of homotopy classes of maps $[Y/H; G/H]$ operates on $[Y^+/H; B_H]$ by $[Y/H; G/H] \xrightarrow{\text{restriction}} [Y^+/H; G/H] \xrightarrow{G/H \hookrightarrow B_H} [Y^+/H; B_H]$. If we combine 2.7 with 2.9 we have

2.11. Theorem: *The strict equivalence classes of fiberings of $(G,X) = (G, G \times_H Y)$ over G/H of type H, where (H,Y) satisfies hypothesis (*) are in one-one correspondence with the image* $[Y/H; G/H] \longrightarrow [Y^+/H; B_H]$.

If $G = T^k$ and H is a closed finite subgroup the analogue of (2.8) when the hypothesis (*) is satisfied becomes

2.12. Corollary: *The strict equivalence classes are in one-one correspondence with the image*

$$\frac{H^1(Y/H; Z^k)}{i\left(H^1(Y/H; Z^k)\right)} \longrightarrow \frac{H^1(Y^+/H; Z^k)}{i\left(H^1(Y^+/H; Z^k)\right)} .$$

In particular, if X is a closed orientable manifold and the set of non-principal orbits is not of codimension 2, then the homomorphism above is bijective..

Proof: The formula above is just a cohomological interpretation of 2.11. To obtain the special case we observe that Y/H is an orientable generalized manifold over the rationals, Q. Since $H^j(Y/H, Y^+/H; Q) \cong \check{H}_{\dim Y - j}(Y - Y^+/H; Q)$ by Poincaré duality, and codimension $(Y - Y^+/H)$ is greater than 2, these Čech homology groups are 0 for $j = 1, 2$. Consequently,

the relative cohomology groups for $j = 1$ and 2 and with integral coefficients are 0 which makes the homomorphism bijective.

Notice that in 2.11 if $G = T^k$ and X/T^k is simply connected then $[X/T^k, T^k/H] = 1$ and hence all fiberings of (T^k, X) over T^k/H of type H are strictly equivalent. Similarly, if $H^1(X/T^k; H) = 0$, all derived actions are strictly equivalent.

3. Fundamental groups of actions derived from a (Z_n, Y)

3.1. Let $(S^1, X) = (S^1, S^1 \times_{Z_n} Y)$ be an equivariant fibering over $(S^1, S^1/Z_n)$ of type Z_n. We wish to describe the fundamental groups arising when derived actions are formed from (Z_n, Y) which has at least one fixed point. We assume that Y is pathwise connected and semi-locally 1-connected. We denote by $T: Y \longrightarrow Y$ the homeomorphism determined by the generator of Z_n. We choose one of the fixed points $y_0 \in Y$ as base point. Since y_0 is fixed the action of Z_n induces a homomorphism

$$Z_n \longrightarrow \mathrm{Aut}(\pi_1(Y, y_0)) .$$

We choose any map $f: Y/Z_n \longrightarrow S^1/Z_n$ such that $f(\nu(y_0)) = \{Z_n\}$. Recall $C_f \subset S^1 \times Y$ is the set of all points (t, y) with $tf(\nu(y)) = \{Z_n\}$. It is more succinct to write S^1/Z_n as the circle S^1, since it is isomorphic to it. That is now, $f: Y/Z_n \longrightarrow S^1$, $f(\nu(y_0)) = 1$ and C_f is the set of points (t, y) with $t^n f(\nu(y)) = 1$. On C_f there are defined two commuting fixed point free homeomorphisms of period n given by

$$T_1(t, y) = (t\lambda^{-1}, Ty)$$

$$T_2(t, y) = (t\lambda, y) ,$$

where $\lambda = \exp(2\pi i/n)$. Thus T_2 induces T_f on $Y_f = C_f/T_1$ and this defines the derived action (Z_n, Y_f). Of course $C_f/T_2 = Y$ and both T_1 and T_1T_2 cover T. We see that $(1, y_0) \in C_f$ and $T_1(1, y_0) = (\lambda^{n-1}, y_0) = T_2^{n-1}(1, y_0)$ since $Ty_0 = y_0$. Thus the T_1 and T_2 orbits of $(1, y_0)$ coincide and hence $[1, y_0] = y_1 \in Y_f$ is fixed under T_f. Thus we obtain a second homomorphism

$$Z_n \longrightarrow \mathrm{Aut}(\pi_1(Y_f, y_1)) .$$

We see now that C_f is an n-fold cyclic covering of both Y and Y_f. Let us use this to define a natural 1-1 correspondence

$$c:\pi_1(Y,y_0) \longrightarrow \pi_1(Y_f,y_1) \quad .$$

Let $\sigma(\tau)$ denote a loop in Y based at y_0. There is a unique path $p(\tau)$ in C_f, covering $\sigma(\tau)$, with $p(0) = (1,y_0)$. But then $p(1) = (\lambda^{-k},y_0) = T_1^k(1,y_0)$ for some $0 \leq k < n$. Thus when $p(\tau)$ is projected into Y_f we again receive a closed loop $c(\sigma(\tau))$ based at y_1. If we regard a base point preserving homotopy of closed loops as a continuous 1-parameter family of closed loops, then we see that $c:\pi_1(Y,y_0) \longrightarrow \pi_1(Y_f,y_1)$ is well defined. The process is completely reversible so that c is a 1-1 correspondence. It is not, however, an isomorphism in general.

We must define a homomorphism $H:\pi_1(y,y_0) \longrightarrow Z_n$. As above a loop $\sigma(\tau)$ in Y at y_0 is covered by a path $p(\tau)$ in C_f issuing from $(1,y_0)$ with $p(1) = (\lambda^{-k},y_0)$. The element $\lambda^{-k} \in Z_n$ depends only on the homotopy class $\sigma \in \pi_1(Y,y_0)$. This defines the required homomorphism. We also receive homomorphisms

$$\alpha:\pi_1(Y,y_0) \longrightarrow \mathrm{Aut}\big(\pi_1(Y,y_0)\big)$$

$$\beta:\pi_1(Y,y_0) \longrightarrow \mathrm{Aut}\big(\pi_1(Y_f,y_1)\big)$$

by $\sigma \longrightarrow \lambda^{-k} \longrightarrow T_*^k$ or $(T_f^{-k})_*$.

Now we can discuss the relation of c to the group structures. Suppose loops $\sigma_1(\tau)$ and $\sigma_2(\tau)$ are covered by paths $p_1(\tau)$, $p_2(\tau)$ both issuing from $(1,y_0)$ with

$$p_1(1) = (\lambda^{-i},y_0)$$

$$p_2(1) = (\lambda^{-j},y_0) \quad .$$

Now,

$$p(\tau) = \begin{cases} p_2(2\tau), & 0 \leq \tau < 1/2 \\ T_2^{-j}p_1(2\tau-1), & 1/2 \leq \tau \leq 1 \end{cases}$$

covers the loop in Y,

$$\sigma(\tau) = \begin{cases} \sigma_2(2\tau), & 0 \leq \tau \leq 1/2 \\ \sigma_1(2\tau-1), & 1/2 \leq \tau \leq 1 \end{cases} \quad ,$$

which represents $\sigma_1 \cdot \sigma_2$ in $\pi_1(Y,y_0)$. (We refer the reader to [6 ; § 2] for our conventions on path multiplication and covering transformations.) However, when the above path $p(\tau)$ is

projected into Y_f we receive a representative of

$$\left((T_f^{-j})_*\left(c(\sigma_1)\right)\right)\cdot c(\sigma_2) \quad .$$

For $\sigma\in\pi_1(Y,y_0)$ let $\beta_\sigma\in \mathrm{Aut}\left(\pi_1(Y_f,y_1)\right)$ be the corresponding automorphism. Then we have

$$c(\sigma_1\sigma_2) = \beta_{\sigma_2}\left(c(\sigma_1)\right)\cdot c(\sigma_2) \ .$$

We must also show that

$$c\circ T_* = (T_f)_*\circ c \ .$$

Again $p(\tau)$ covers $\sigma(\tau)$, and $T\sigma(\tau)$ is covered by $T_2T_1p(\tau)$, but this covers $T_f c\left(\sigma(\tau)\right)$ in Y_f, hence $c\circ T_* = (T_f)_*\circ c$. Remember that if $H(\sigma)=\lambda^{-k}$ then $\alpha_\sigma = T_*^k$, $\beta_\sigma = (T_f^{-k})_*$, hence $\beta_\sigma\circ c\circ\alpha_\sigma = c$ for all $\sigma\in\pi_1(Y,y_0)$. In particular, if we define a new product in $\pi_1(Y,y_0)$ by

$$\sigma_1 * \sigma_2 = \alpha_{\sigma_2}(\sigma_1)\cdot\sigma_2$$

we have

$$c(\sigma_1*\sigma_2) = c\left(\alpha_{\sigma_2}(\sigma_1)\cdot\sigma_2\right) = \left(\beta_\sigma c\alpha_{\sigma_2}(\sigma_1)\right)\cdot c(\sigma_2) = c(\sigma_1)\cdot c(\sigma_2) \quad .$$

Thus, if we can show that this new product is a group structure then we shall have a convenient isomorphic copy of $\pi_1(Y_f,y_1)$. The only point needing verification is associativity. This will follow if we can prove the identity

$$\alpha_{\alpha_{\sigma_2}(\sigma_1)} \equiv \alpha_{\sigma_1}$$

for all σ_1,σ_2 in $\pi_1(Y,y_0)$. Suppose $\alpha_{\sigma_2} = T_*^i$, $\alpha_{\sigma_1} = T_*^j$, then $\sigma_1(\tau)$ is covered by $p_1(\tau)$ issuing from $(1,y_0)$ with $p_1(1) = (\lambda^{-j},y_0)$. Now $(T_1T_2)^i p(\tau)$ also issues from $(1,y_0)$ and covers $T^i\sigma_1(\tau)$, but $(T_1T_2)^i(\lambda^{-j},y_0) = (\lambda^{-j},y_0)$. Hence

$$\alpha_{T^i(\sigma_1)} = \alpha_{\alpha_{\sigma_2}(\sigma_1)} = \alpha_{\sigma_1} \quad .$$

Associativity now is

$$(\sigma_1 * \sigma_2) * \sigma_3 = \left(\alpha_{\sigma_2}(\sigma_1)\cdot\sigma_2\right) * \sigma_3$$
$$= \alpha_{\sigma_3}\left(\alpha_{\sigma_2}(\sigma_1)\cdot\sigma_2\right)\cdot\sigma_3$$
$$= \left(\alpha_{\sigma_3}\left(\alpha_{\sigma_2}(\sigma_1)\right)\right)\cdot\left(\alpha_{\sigma_3}(\sigma_2)\right)\cdot\sigma_3$$

$$\sigma_1 * (\sigma_2 * \sigma_3) = \sigma_1 * \left(\alpha_{\sigma_3}(\sigma_2)\cdot\sigma_3\right)$$
$$= \alpha_{\alpha_{\sigma_3}(\sigma_2)\cdot\sigma_3}(\sigma_1)\cdot\alpha_{\sigma_3}(\sigma_2)\cdot\sigma_3$$
$$= \left(\alpha_{\sigma_3}\left(\alpha_{\sigma_2}(\sigma_1)\right)\right)\cdot\alpha_{\sigma_3}(\sigma_2)\cdot\sigma_3$$

since $\alpha_{\alpha_{\sigma_3}(\sigma_2)\cdot\sigma_3} = \alpha_{\alpha_{\sigma_3}(\sigma_2)} \circ \alpha_{\sigma_3} = \alpha_{\sigma_3} \circ \alpha_{\alpha_{\sigma_3}(\sigma_2)} = \alpha_{\sigma_3} \circ \alpha_{\sigma_2}$.

There is an alternative, and more direct, description of the homomorphism $\pi_1(Y, y_0) \longrightarrow \mathrm{Aut}\left(\pi_1(Y, y_0)\right)$ and hence of the new group structure. The composite map $f\nu: (Y, y_0) \longrightarrow (S^1, 1)$ induces a homomorphism

$$f_*\nu_*: \pi_1(Y, y_0) \longrightarrow \pi_1(S^1, 1)$$

and there is

$$\pi_1(S^1, 1) \longrightarrow Z_n \longrightarrow \mathrm{Aut}\left(\pi_1(Y, y_0)\right).$$

In this way $\sigma \longrightarrow \alpha_\sigma \in \mathrm{Aut}\left(\pi_1(Y, y_0)\right)$. For any $\sigma \in \pi_1(Y, y_0)$ we see that $\nu_*(\sigma) = \nu_*\left(T_*(\sigma)\right)$ and this yields $\alpha_{\alpha_{\sigma_2}(\sigma_1)} = \alpha_{\sigma_1}$. That is $\nu_*\left(\alpha_{\sigma_2}(\sigma_1)\right) = \nu_*(\sigma_1)$. Thus we can define the new group law by $\sigma_1 * \sigma_2 = \alpha_{\sigma_2}(\sigma_1)\cdot\sigma_2$ as before.

Let us call π_f this new group. It is $\pi_1(Y_f, y_1)$. Note that $\nu_*: \pi_f \longrightarrow \pi_1(Y/Z_n)$ is a group homomorphism. Since y_0 is fixed it is a theorem of Floyd that

$$\nu_*: \pi_f \longrightarrow \pi_1(Y/Z_n)$$

is an epimorphism. The kernel $K \subset \pi_f$ is easily seen in subgroup structure to be independent of f. Thus we see that we have a group extension

$$(3.2) \qquad 1 \longrightarrow K \longrightarrow \pi_f \longrightarrow \pi_1(Y/Z_n) \longrightarrow 1 \ ,$$

for each $f: Y/Z_n \longrightarrow S^1$. Each π_f can be realized as the fundamental group of the corresponding derived action.

3.3. An elementary, but useful illustration is the following. Let (Z_n, M^k) be an effective action with at least one fixed point on a closed aspherical manifold. From our previous work [6] we know $Z_n \longrightarrow \mathrm{Aut}(\pi_1(M^k))$ is a monomorphism. Let $Y = M^k \times S^1$ and let Z_n act only on the first factor so that there is still a fixed point y_0. Then $(M^k \times S^1)/Z_n = M^k/Z_n \times S^1$, so take f to be the projection map onto S^1. Now

$$\pi_1(Y, y_0) = \pi_1(M^k) \oplus \pi_1(S^1) \quad .$$

After some thought it will be seen that in the resulting derived action $\pi_1(Y_f)$ is the semi-direct product $\pi_1(M^k) \circ Z$. The required homomorphism $Z \longrightarrow \mathrm{Aut}(\pi_1(M^k))$ is the composition $Z \longrightarrow Z_n \longrightarrow \mathrm{Aut}(\pi_1(M^k))$. Of course Y_f is still a closed aspherical manifold.

Some immediate examples come to mind: Take $M^k = S^1$ and (Z_2, S^1) to be $z \longrightarrow \bar{z}$. Then $Y = S^1 \times S^1$ and $Y_f = S^1 \times_{Z_2} S^1$ = Klein bottle. This has an obvious generalization $M^k = T^k = S^1 \times \ldots \times S^1$, and $(z_1, \ldots, z_k) \longrightarrow (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_k)$. Then $Y = T^{k+1}$, $Y_f = S^1 \times_{Z_2} T^k$ which is a flat manifold with non-abelian fundamental group.

3.4. Corollary: If (Z_n, M^k) is an effective action on a closed aspherical manifold then associated to $(Z_n, S^1 \times M^k) = (Z_n, Y)$, which acts trivially on the first factor, is the derived action $(Z_n, Y_f) = (Z_n, S^1 \times_{Z_n} M^k)$ where $\pi_f = \pi_1(M^k, *) \circ Z$.

Furthermore, (Z_n, Y_f) can always be extended to an action of S^1 on Y_f and consequently $(S^1, X) = (S^1, S^1 \times_{Z_n} Y) = (S^1, S^1 \times_{Z_n} Y_f)$ is homeomorphic to $S^1 \times Y_f$.

To see that the action (Z_n, Y_f) can always be extended to an S^1-action, note that Y_f can be considered as triples

$$\left[t, x, t^n\right] \quad .$$

Here $\left[t, x, t^n\right] = \left[t\lambda^{-1}, Tx, t^n\right]$, and $T_f\left[t, x, t^n\right] = \left[t\lambda, x, t^n = (t\lambda)^n\right]$. But then (S^1, Y_f) is

$$z\left[t, x, t^n\right] = \left[zt, x, (zt)^n\right] \quad .$$

Note $\lambda\left[t, x, t^n\right] = \left[t\lambda, x, t^n\right]$ so this extends (Z_n, Y_f). Since the generating homeomorphism T_f is isotopic to the identity in S^1, the total space of the fiber bundle $(S^1, S^1 \times_{Z_n} Y_f)$ is homeomorphic to $S^1 \times Y_f$ by enlarging the structure group.

3.5. The reader may be interested in pursuing the case when G is other than the circle and H is a finite normal subgroup of G. The analysis of this section largely carries over to the general case.

3.6. Let us now turn to a special situation which is really included in the preceding. This will be applied when (T, Y) is homotopic to the identity. (This was the case for (T_f, Y_f) in 3.4.) We shall begin first algebraically and then discuss the topological significance. The first two lemmas are repetitions of part of 3.1.

Fix a group G together with a homomorphism $N: G \longrightarrow Z$. We choose an element $\beta \in G$ and define a new composition rule

$$g * h = \beta^{N(h)} g \beta^{-N(h)} h \ .$$

3.7. Lemma: $N(gh) = N(hg) = N(g*h) = N(h*g) = N(g)+N(h)$.

Proof: Trivial since Z is abelian.

3.8. Lemma: The new composition rule defines a group structure, denoted by G_*, on the set G.

1) Associativity:

$$\begin{aligned}(g*h)*k &= (\beta^{N(h)} g \beta^{-N(h)} h)*k \\ &= \beta^{N(k)+N(h)} g \beta^{-N(h)} h \beta^{-N(k)} k \\ &= \beta^{N(h*k)} g \beta^{-N(h*k)} \beta^{N(k)} h \beta^{-N(k)} k \\ &= g*(h*k) \ .\end{aligned}$$

2) The identity of G is also the identity of G_*

3) Inverses:

$$N(\beta^{-N(g)} g^{-1} \beta^{N(g)}) = -N(g) \qquad \text{so that}$$

$$g*(\beta^{-N(g)} g^{-1} \beta^{N(g)}) = \beta^{-N(g)} g \beta^{N(g)} \beta^{-N(g)} g^{-1} \beta^{N(g)} = e \ .$$

3.9. Lemma: There is a canonical homorphism

$$c: G_* \longrightarrow G$$

given by $c(g) = \beta^{-N(g)} g$.

Proof: $c(g * h) = \beta^{-N(gh)}(g * h)$

$$= \beta^{-N(gh)}\beta^{N(h)}g\beta^{-N(h)}h$$

$$= \beta^{-N(g)}g\beta^{-N(h)}h = c(g)c(h) .$$

3.10. Theorem: The kernel of c is trivial unless $N(\beta) = 1$, in which case it is the free cyclic group generated by $\beta \in G_*$. The image of c is the kernel of the homomorphism

$$G \longrightarrow Z/(1 - N(\beta))Z$$

obtained by forming the composition of N with the quotient homomorphism $Z \longrightarrow Z/(1 - N(\beta))Z$.

Proof: Suppose $c(g) = \beta^{-N(g)}g = e$, then $N(g) = N(g)N(\beta)$. If $N(\beta) = 1$ it follows $N(g) = 0$, but then $\beta^0 g = e$ implies $e = g$. If $N(\beta) = 1$ then $c(\beta^m) = e$ for all $m \in Z$. Now $\beta * \beta = \beta\beta\beta^{-1}\beta = \beta^2$ so by induction we see that the kernel of c, if $N(\beta) = 1$, is the cyclic group generated by $\beta \in G_*$.

Now suppose $N(h) = m\big(1 - N(\beta)\big)$ for some $m \in Z$. Let $g = \beta^m h$ so that $N(g) = mN(\beta) + N(h) = m$. Then $c(g) = h$. Conversely $N(\beta^{-N(g)}g) = -N(g)N(\beta) - N(g) = N(g)\big(1 - N(\beta)\big)$. Observe $c: G_* \simeq G$ if and only if $N(\beta) = 0$ or 2.

3.11. Corollary: If $N(\beta) = 1$ then $G_* \simeq Z \times K$ where $K \subset G$ is the kernel of $N: G \longrightarrow Z$.

We note that $N: G_* \longrightarrow Z$ is still an epimorphism. Then $g \longrightarrow \big(N(g), c(g)\big)$ is the required isomorphism.

3.12. The algebraic construction discussed here arises in the following topological situation. Suppose (Z_n, Y) is a cyclic transformation group with at least one fixed point and suppose that the homeomorphism $T: Y \longrightarrow Y$ corresponding to a generator of Z_n is homotopic as a map to the identity. We thus have a map

$$H: Y \times I \longrightarrow Y$$

such that $H(y, 0) \equiv y$, $H(y, 1) \equiv T(y)$. Select a point y_0 fixed under T. Then we obtain a closed loop $\beta(t) = H(y_0, t)$ at y_0 and this represents $\beta \in \pi_1(Y, y_0)$. Since y_0 is fixed there is an automorphism $T_*: \pi_1(Y, y_0) \longrightarrow \pi_1(Y, y_0)$ induced by T. We have shown that $T_*(\sigma) = \beta\sigma\beta^{-1}$ in this case. We are interested in derived actions (Z_n, Y_f) arising from (Z_n, Y). These are derived from maps $f: Y/Z_n \longrightarrow S^1$, for which we may assume without loss

of generality that $f(\nu(y_0)) = 1$, where $\nu: Y \longrightarrow Y/Z_n$ is the quotient map. Naturally this f induces $f_*\nu_*: \pi_1(Y, y_0) \longrightarrow \pi_1(S^1, 1) \simeq Z$. Using the specified β and this homomorphism we apply the $*$-construction to $\pi_1(Y, y_0)$ and obtain a group isomorphic to $\pi_1(Y_f, y_1)$, the fundamental group for the derived action.

Because T is homotopic to the identity, $\nu^*: H^1(Y/Z_n; Z) \simeq H^1(Y; Z) \simeq \mathrm{Hom}(\pi_1(Y, y_0), Z)$. Hence any homomorphism $N: \pi_1(Y, y_0) \longrightarrow Z$ can be induced by a map of the form

$$Y \xrightarrow{\nu} Y/Z_n \xrightarrow{f} S^1$$

The β uniquely determines a homology class $[\beta] \in H_1(Y; Z)$ and

$$N(\beta) = \langle \nu^* f^*(i), [\beta] \rangle \in Z .$$

If $N(\beta) = 1$, then $\pi_1(Y_f, y_1)$ is a direct product of Z with the kernel of N. If $N(\beta) \neq 1$, then $\pi_1(Y_f, y_1)$ is isomorphic to a normal subgroup of $\pi_1(Y, y_0)$ whose quotient lies in $Z/(1 - N(\beta))Z$.

We can see all possibilities as follows. Begin with an action (Z_{nk}, W) for which the subgroup Z_n has a fixed point, say w_0. Let $\lambda = \exp(2\pi i/nk)$ and form as usual $Y = S^1 \times_{Z_{nk}} W$ writing $((t, w)) = ((t\lambda^{-1}, \lambda w))$ for a point in Y. Let (Z_n, Y) be given by $T((t, w)) = ((t\lambda^k, w)) = ((t, \lambda^k w))$. Of course there is a fixed point $y_0 = ((t, w_0))$. This action (Z_n, Y) can be extended to an action of S^1 on Y. We can choose the $\beta \in \pi_1(Y, y_0)$ so that β^n is the image of the generator of $\pi_1(S^1, 1) \longrightarrow \pi_1(Y, y_0)$ under the map $t \longrightarrow ty_0$. There is the map $\phi((t, w)) = t^{nk}$, which factors into

$$Y \xrightarrow{\nu} Y/Z_n \xrightarrow{f} S^1 .$$

Of course $\phi_*: \pi_1(Y; Z) \longrightarrow \pi_1(S^1)$ is an epimorphism with kernel $\pi_1(W)$ since ϕ is a fibration. Further, $\phi_*(\beta) = k$ since $\phi_*(\beta^n) = nk$ the image of the generator of $\pi_1(S^1)$ under the map $t \longrightarrow t^{nk}$. Thus if $k = 1$, $\pi_1(Y_\phi, y_1) \simeq Z \times \pi_1(W, w_0)$.

The reader might wish to select (Z_2, S^1) given by $t \longrightarrow \bar{t}$ and see Y is the Klein bottle while Y_ϕ is T^2. Another instructive example with $n = 2$, $k = 3$ is found by letting Z_6 act on the curve $S \subset CP(2)$ given by

$$z_1^6 + z_2^6 + z_3^6 = 0 \quad .$$

The action is $[z_1, z_2, z_3] \longrightarrow [\lambda^2 z_1, \lambda^3 z_2, z_3]$. The fixed point set of $Z_2 \subset Z_6$ is the finite set $\left\{\left[1, 0, \exp\left(\frac{2\pi ij}{12}\right)\right]\right\}$, $1 \leq j < 12$, j odd.

4. Cyclic quotients of fibered actions: $X(q)$

Consider an action (Z_n, Y) on a pathwise connected space. Let $\lambda = \exp(2\pi i/n)$ and let $T: Y \to Y$ be the periodic homeomorphism corresponding to this generator. For any integer q, $0 < q < n$, and $(q,n) = 1$ we introduce the space $X(q)$ obtained from $S^1 \times Y$ by the identification $(t,y) \sim (t\lambda^{-j}, T^{qj}(y))$, $0 \le j < n$. A point in $X(q)$ is denoted by $((t,y))_q$.

4.1 Theorem: For any such choice of q, $S^1 \times X(1)$ is homeomorphic to $S^1 \times X(q)$.

Proof: Fix q and choose the integer m with least absolute value satisfying $qm \equiv 1$ mod n. This exists since $(q,n) = 1$. Define $(Z_n, X(1))$ by

$$\tilde{T}_q((t,y))_1 = ((t\lambda^q, y))_1 = ((t, T^q y))_1$$

and $(Z_n, X(q))$ by

$$\tilde{T}_1((t,y))_q = ((t\lambda^m, y))_q = ((t, T^{qm} y))_q = ((t, Ty))_q \ .$$

We immediately note that both $(Z_n, X(1))$ and $(Z_n, X(q))$ can be extended to actions of S^1; that is, for $\tau \in S^1$

$$\tau((t,y))_1 = ((t\tau \ , y))_1$$

$$\tau((t,y))_q = ((t\tau^m, y))_q \ .$$

Thus we may form the fiber bundles

$$\left(S^1 \times_{Z_n} X(1), \quad X(1), \quad S^1/Z_n, \quad Z_n\right)$$

$$\left(S^1 \times_{Z_n} X(q), \quad X(q), \quad S^1/Z_n, \quad Z_n\right)$$

for which the projections:

$$\left(S^1, S^1 \times_{Z_n} X(1)\right) \xrightarrow{X(1)} (S^1, S^1/Z_n)$$

$$\left(S^1, S^1 \times_{Z_n} X(q)\right) \xrightarrow{X(q)} (S^1, S^1/Z_n)$$

are equivariant. The action of Z_n on $X(1)$ and $X(q)$ is generated by the periodic homeomorphisms $\tilde{T}_q$ and $\tilde{T}_1$. Since each of these homeomorphisms is isotopic to the identity

the fiber bundles are trivial when their structure groups are enlarged to S^1, $Z_n \subseteq S^1$. Hence we have that

$$S^1 \times_{Z_n} X(1) = S^1 \times X(1)$$

$$S^1 \times_{Z_n} X(q) = S^1 \times X(q) \ ,$$

where equality means "bundle isomorphic to".

To prove that $S^1 \times_{Z_n} X(1)$ is homeomorphic to $S^1 \times_{Z_n} X(q)$ we may introduce $(Z_n + Z_n, S^1 \times S^1 \times Y)$. Let

$$T_1(t_1, t_2, y) = (t_1, t_2\lambda^{-1}, Ty)$$

$$T_q(t_1, t_2, y) = \left(t_1\lambda^{-1}, t_2, T^q(y)\right) \ .$$

These commute and each has period n, hence the required action is defined. If the Z_n subgroup generated by T_1 is first factored out we obtain $S^1 \times X(1)$, on which T_q induces

$$\left(t_1, ((t_2, y))_1\right) \longrightarrow \left(t_1\lambda^{-1}, ((t_2, T^q(y)))_1\right)$$

so that $(S^1 \times S^1 \times Y)/Z_n + Z_n = S^1 \times_{Z_n} X(1)$. Similarly if T_q is factored out first we receive $S^1 \times X(q)$, on which T_1 induces $\lambda^{-1} \times \tilde{T}_1$ so that $(S^1 \times S^1 \times Y)/Z_n + Z_n = S^1 \times_{Z_n} X(q)$ also.

4.2 The spaces $X(1)$ and $X(q)$ are related as follows. Let us introduce $\rho = \exp(2\pi i/nm)$ so that $\rho^m = \lambda$. On $X(1)$ we define an action of the cyclic group Z_{nm} by

$$((t, y))_1 \longrightarrow \left(\left(t\rho^{-1}, T^q(y)\right)\right)_1 \ .$$

We assert that $X(1)/Z_{nm} = X(q)$. This is seen by defining a map

$$\nu : X(1) \longrightarrow X(q)$$

where $\nu((t, y))_1 = ((t^m, y))_q$. Note that

$$\nu\big((t\lambda^{-1}, Ty)\big)_1 = \big((t^m\lambda^{-m}, Ty)\big)_q = \big((t^m\lambda^{-m}, T^{qm}TT^{-qm}y)\big)_q$$

$$= \big((t^m, T^{1-qm}y)\big)_q = \big((t^m, y)\big)_q$$

since $1 - qm \equiv 0 \bmod p$. Thus ν is well defined. Furthermore, $\nu\big((t\rho^{-1}, T^q(y)\big)_1 = \big((t^m\lambda^{-1}, T^q(y)\big)_q = \big((t^m, y)\big)_q$. Hence ν does induce a map $X(1)/Z_{nm} \longrightarrow X(q)$. Finally, suppose $\nu\big((t,y)\big)_1 = \big((t^m, y)\big)_q = \big((\tau^m, z)\big)_q = \nu\big((\tau, z)\big)_q$. Then for some j, $0 \le j < n$ we have

$$\tau^m = t^m\rho^{-mj}$$

$$z = T^{qj}(y) \quad .$$

Thus $(\tau/t)^m = (\rho^{-j})^m$. Hence τ/t is ρ^{-j} multiplied by some m^{th} root of unity; that is, by ρ^{-nr} for some $0 \le r < m$. But since T^n = identity we have $\tau = t\rho^{-j-nr}$, $z = T^{q(j+nr)}y$. Thus, $\big((t\rho^{-j-nr}, T^{q(j+nr)}(y)\big)_1 = \big((\tau, z)\big)_1$. Hence ν induces a homeomorphism

$$X(1)/Z_{nm} \longrightarrow X(1)$$

In a similar vein $X(1)$ is the quotient of an action of Z_{nq} on $X(q)$.

4.3 In the remaining part of the paper we shall, in effect, investigate algebraically the problem of when $X(1)$ may not be homeomorphic to $X(q)$. Thus, it will be extremely useful in terms of motivation and algebraic insight to build into the previous geometric constructions as much symmetry as is possible. We shall now re-examine 4.1 and 4.2 with this in mind.

We shall use the subscripts on the circles S^1_1 and S^1_2 and the superscripts on the cyclic groups $Z^{(1)}_n$ and $Z^{(2)}_n$ for the purposes of identification.

We begin with $S^1_1 \times S^1_2 \times Y$ and define an action of $S^1_1 \times S^1_2 \times Z^{(1)}_n \times Z^{(2)}_n$ on this space. The action will not be effective since there will be a subgroup isomorphic to Z_n which fixes everything. However, each of the factors will act freely. The circle groups act on the corresponding circle factors by translation and we define the actions of $Z^{(1)}_n$ and $Z^{(2)}_n$ by the previous formulae

$$T_1(t_1, t_2, y) = (t_1, t_2\lambda^{-1}, Ty)$$

$$T_q(t_1, t_2, y) = (t_1\lambda^{-1}, t_2, T^q y) \;.$$

We obtain a toral action

$$\left(S_1^1 \times S_2^1,\ (S_1^1 \times S_2^1 \times Y)/\ Z_n^{(1)} \times Z_n^{(2)}\right) \quad .$$

But since the actions of S_1^1, S_2^1 and $Z_n^{(1)}$ and $Z_n^{(2)}$ all commute, we write their quotient of $S_1^1 \times S_2^1 \times Y$ by $Z_n^{(1)} \times Z_n^{(2)}$ in alternative ways:

$$\left(S_1^1 \times S_2^1,\ S_1^1 \times_{Z_n^{(2)}} (S_2^1 \times_{Z_n^{(1)}} Y)\right) = \left(S_1^1 \times S_2^1,\ S_1^1 \times_{Z_n^{(2)}} X(1)\right)$$

$$= \left(S_1^1 \times S_2^1,\ S_2^1 \times_{Z_n^{(1)}} (S_1^1 \times_{Z_n^{(2)}} Y)\right) = \left(S_1^1 \times S_2^1,\ S_2^1 \times_{Z_n^{(1)}} X(q)\right) \quad .$$

<u>Thus</u>, we immediately see that $S_1^1 \times_{Z_n^{(2)}} X(1)$ <u>and</u> $S_2^1 \times_{Z_n^{(1)}} X(q)$ <u>are</u> $(S_1^1 \times S_2^1)$-<u>equivariantly homeomorphic</u>.

We must now show that the action of $Z_n^{(2)}$ induced on $S_2^1 \times_{Z_n^{(1)}} Y$ is contained within the S_2^1 action induced on $S_2^1 \times_{Z_n^{(1)}} Y$. Hence $\left(S_1^1 \times S_2^1,\ S_1^1 \times_{Z_n^{(2)}} (S_1^1 \times_{Z_n^{(1)}} Y)\right)$, which may be fibered over $S_1^1 / Z_n^{(2)}$ with fiber $X(1) = S_2^1 \times_{Z_n^{(1)}} Y$, may have its structure group enlarged to the connected group S_2^1. Consequently, $S_1^1 \times_{Z_n^{(2)}} X(1)$ is homeomorphic to $S_1^1/Z_n^{(2)} \times X(1) = S^1 \times X(1)$. Let us denote a point in $S_1^1 \times (S_2^1 \times_{Z_n^{(1)}} Y)$ by $\left(t_1, ((t_2, y))_1\right)$. The action of $Z_n^{(2)}$ on $(S_2^1 \times_{Z_n^{(1)}} Y)$ is given by

$$((t_2, y))_1 \xrightarrow{\ \tilde{T}_q\ } ((t_2, T_y^q))_1 = ((t_2 \lambda^q, y))_1 \quad .$$

But, the action of S_2^1 on $(S_2^1 \times_{Z_n^{(1)}} Y)$ is given by

$$\tau_2((t_2, y))_1 = ((\tau_2 t_2, y))_1 .$$

Hence, $((t_2\lambda^q, y))_1 = \lambda^q((t_2, y))_1$, $\lambda^q \in S_2^1$.

Similarly, we show that the $Z_n^{(1)}$ action induced on $S_1^1 \times_{Z_n^{(2)}} Y$ is contained within the S_1^1 action induced on $S_1^1 \times_{Z_n^{(2)}} Y$. Hence, $\left(S_2^1, S_2^1 \times_{Z_n^{(1)}} (S_1^1 \times_{Z_n^{(2)}} Y)\right) =$ $\left(S_2^1, S_2^1 \times_{Z_n^{(1)}} X(q)\right)$, which fibers equivariantly over $(S_2^1, S_2^1/Z_n^{(1)})$, is homeomorphic to $\left(S_2^1/Z_n^{(1)} \times X(q)\right) = S^1 \times X(q)$.

In conclusion, we have toral actions on $S^1 \times X(1)$ and $S^1 \times X(q)$ and an equivariant homeomorphism between them.

4.4. As one has probably noticed in 4.2, the actions of Z_{nm} on $X(1)$ and Z_{nq} on $X(q)$ are effectively actions of Z_m and Z_q, respectively. The connections between $X(1)$ and $X(q)$ may be put into a slightly different perspective as follows. We use the same notation as employed earlier.

Define $(S^1 \times Z_{nm}, S^1 \times Y)$ by

$$(t, y) \xrightarrow{J} (t\rho^{-1}, T^q y) ,$$

and the action of S^1 by translation. The cyclic group Z_{nm} is generated by J. J^m generates a cyclic group isomorphic to Z_n. Since

$$(t, y) \xrightarrow{J^m} (t\lambda^{-1}, Ty) ,$$

the action of the group generated by J^m is the same as the action of $Z_n^{(1)}$. The action of the complementary group Z_m, generated by J^n, is given by

$$(t, y) \xrightarrow{J^n} (t\rho^{-n}, T^{qn} y) = (t\rho^{-n}, y) .$$

Thus the action of Z_m is embedded in the translation action of S^1. We then observe that

$$\bar{\nu}:(t, y) \longrightarrow (t^m, y) \in (S^1/Z_m) \times Y$$

is just the orbit map of Z_m. There is induced an action of S^1/Z_m on $(S^1/Z_m) \times Y$ which commutes, of course, with $Z_{mn}/Z_m \cong Z_n$. Notice that the action of Z_n generated by J^m on $(S^1/Z_m) \times Y = (S^1 \times Y)/Z_m$ is given by

$$\begin{array}{ccc}
(t,y) & \xrightarrow{\bar{\nu}} & (t^m, y) \\
\downarrow{\scriptstyle J^m} & & \downarrow{\scriptstyle J^m} \\
(t\lambda^{-1}, Ty) & \xrightarrow{\bar{\nu}} & (t^m\lambda^{-m}, T^{qm}y) = (t^m\lambda^{-m}, (T^q y)^m) .
\end{array}$$

But this action of Z_n on $S^1/Z_m \times Y = S^1 \times Y$ is equivalent to the action of $Z_n^{(2)}$ on $S^1 \times Y$. In fact, q-times the generator J^m is also a generator of Z_{mn}/Z_m and the action of $(J^m)^q$ is exactly the action of the earlier chosen generator of $Z_n^{(2)}$.

In terms of the S^1-actions we have

$$\begin{array}{ccc}
(S^1, S^1 \times Y, Z_{mn}) & \xrightarrow[\bar{\nu}]{/Z_m} & (S^1/Z_m, S^1/Z_m \times Y, Z_n) \\
\downarrow{\scriptstyle /Z_n} & & \downarrow{\scriptstyle /Z_n} \\
(S^1, S^1 \times_{Z_n^{(1)}} Y, Z_m) & \xrightarrow[\nu]{/Z_m} & (S^1/Z_m, S^1/Z_m \times_{Z_n^{(2)}} Y).
\end{array}$$

We have identified $(S^1, S^1 \times_{Z_n^{(1)}} Y, Z_m)$ with $(S^1, X(1), Z_m)$ and shown that the Z_m action is embedded in the S^1 action. Thus, $(S^1/Z_m, X(1)/Z_m)$ is identified, via ν, with $(S^1, X(q))$.

Similarly, we may define on $S^1 \times Y$ an action of Z_{nq} so that

$$(S^1, S^1 \times Y/Z_n, Z_q) = (S^1, S^1 \times_{Z_n^{(2)}} Y, Z_q)$$

and the Z_q action is embedded in the S^1-action. We just put $\sigma = \exp(2\pi i/qn)$ and define

$$\begin{array}{ccc}
(t,y) & \xrightarrow[\bar{\mu}]{} & (t^q, y) \\
\downarrow & & \downarrow \\
(t\sigma^{-1}, Ty) & \xrightarrow[\bar{\mu}]{} & (t^q\sigma^{-q}, T^q y)
\end{array}$$

We receive a similar

$$\begin{tikzcd}
(S^1, S^1 \times Y, Z_{nq}) \arrow[r, "/Z_q", "\bar{\mu}"'] \arrow[d, "/Z_n"] & (S^1/Z_q, S^1/Z_q \times Y, Z_n) \arrow[d, "/Z_n"] \\
(S^1, X(q)) \arrow[r, "/Z_q", "\mu"'] & (S^1/Z_q, S^1/Z_q \times_{Z_n^{(1)}} Y)
\end{tikzcd}$$

If P denotes the generator of Z_{nq}, then the induced action of $Z_{nq}/Z_q \cong Z_n$ on $S^1/Z_q \times Y$ is equivalent to the action of $Z_n^{(1)}$. (We choose the generator $(P^q)^m$ of Z_n.) We summarize the discussion in the following.

4.5 Theorem: For $Z_m \subseteq S^1$, and $Z_q \subseteq S^1$ there are equivariant homeomorphisms:

$$(S^1/Z_m, X(1)/Z_m) \approx (S^1, X(q)) \quad , \quad \underline{\text{and}}$$

$$(S^1/Z_q, X(q)/Z_q) \approx (S^1, X(1)).$$

Furthermore, the actions $(X(1), Z_m)$ and $(X(q), Z_q)$ are free.

To see that Z_m and Z_q are free actions we may check directly. Alternatively, we simply observe that since m and q are relatively prime to n all the stability groups of $(S^1, X(1))$ and $(S^1, X(q))$ have orders which are divisors of n.

4.6. Finally, we give still other explicit homeomorphisms. Let $1 - qm = nr$, define

$$S^1 \times X(1) \longrightarrow S^1 \times X(q)$$

by

$$\left(t_1, ((t_2, y))_1\right) \longrightarrow \left(t_2^n t_1^{-q}, ((t_2^m t_1^r, y))_q\right),$$

and

$$S^1 \times X(q) \longrightarrow S^1 \times X(1)$$

by

$$\left(\tau_1, ((\tau_2, y))_q\right) \longrightarrow \left(\tau_2^n \tau_1^{-m}, ((\tau_2^q \tau_1^r, y))_1\right).$$

One simply checks these are inverses to each other. Also note that we have natural actions of $S_1^1 \times S_2^1$ on $S^1 \times X(1)$ and $S^1 \times X(q)$, with S_1^1 acting by translation on the first factor and S_2^1 on the second factor. If we perform the automorphisms $S_1^1 \times S_2^1 \longrightarrow S_1^1 \times S_2^1$ defined by

$$(t_1, t_2) \longrightarrow (t_1^{-q} t_2^n,\ t_1^r t_2^m)\,, \qquad \text{and}$$

$$(\tau_1, \tau_2) \longrightarrow (\tau_1^{-m} \tau_2^n,\ \tau_1^r \tau_2^q)$$

respectively, the homeomorphisms are equivariant with respect to these automorphisms.

5. Algebraic preliminaries

5.1. In the preceding section we have discussed part of the underlying topological situation. Our chief resource for producing interesting examples where $X(1)$ is not homeomorphic to $X(q)$ is the fundamental group. We shall eventually develop several methods which will enable us to detect, for interesting spaces, that $\pi_1(X(1)) \neq \pi_1(X(q))$. Thus, we devote the next 3 sections to the study of the corresponding group theoretical questions.

We begin with a group extension

$$1 \longrightarrow \pi \xrightarrow{\mu} N \xrightarrow{\nu} Z_n \longrightarrow 0$$

and we take $a \in H^2(N;Z)$ to be the image under $\nu^*: H^2(Z_n; Z) \longrightarrow H^2(N; Z)$ of the extension

$$1 \longrightarrow Z \xrightarrow{n} Z \xrightarrow{\sigma} Z_n \longrightarrow 0\,.$$

For each integer m, with $(m,n) = 1$, we consider the central extension corresponding to ma; that is,

$$0 \longrightarrow Z \longrightarrow \mathcal{L}_m \longrightarrow N \longrightarrow 1.$$

Since $na = 0$ we see immediately that $\mathcal{L}_m = \mathcal{L}_{m+n}$. Now the automorphism of Z_n given by multiplication with m yields on $H^2(Z_n; Z)$ multiplication by m again as the induced automorphism. Thus $\mathcal{L}_m \subset N \times Z$ is the subgroup of all pairs (α, s) with $m(\nu(\alpha)) = \sigma(s)$.

5.2. Lemma: If $(m,n) = 1$, then $\mathcal{L}_m \cong \mathcal{L}_{-m}$.

Proof: The isomorphism is given by $(\alpha, s) \to (\alpha, -s)$.

More generally, we can prove a stable isomorphism theorem.

5.3. Theorem: If $(m,n) = 1$ then

$$\mathcal{L}_1 \times Z \simeq \mathcal{L}_m \times Z .$$

Proof: There is a pair of integers q and r such that $mq+nr = 1$, so that $\begin{pmatrix} m & -n \\ r & q \end{pmatrix} \in SL(2,Z)$, (cf. 4.6). An element of $L_1 \times Z$ is a triple (α, s, t) with $\nu(\alpha) = \sigma(s)$. The isomorphism is then given by $(\alpha, s, t) \longrightarrow (\alpha,\ ms-nt,\ rs+qt)$. Note that

$$\sigma(ms-nt) = m\sigma(s) = m\nu(\alpha) .$$

The inverse is explicitly given by $\begin{pmatrix} q & n \\ -r & m \end{pmatrix}$ of course.

It is not generally true, however, that $\mathcal{L}_1 \simeq \mathcal{L}_m$. We propose to investigate this question algebraically, and later to interpret topologically.

For each m the central extension $0 \longrightarrow Z \longrightarrow \mathcal{L}_m \longrightarrow N \longrightarrow 1$ is given by $k \longrightarrow (e, nk)$, where $e \in N$ is the identity.

5.4. Lemma: If the center of N is trivial then the center of $\mathcal{L}_m$ is the image of $Z \longrightarrow \mathcal{L}_m$.

This is clear, and it immediately follows that

5.5 Theorem: If the center of N is trivial then $\mathcal{L}_m \simeq \mathcal{L}_1$ if and only if there is an automorphism $\phi: N \longrightarrow N$ for which

$$\phi^*(a) = \pm ma .$$

Proof. Since N is centerless, and $\psi: \mathcal{L}_m \simeq \mathcal{L}_1$ must preserve centers, we obtain a commutative diagram with vertical isomorphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & \mathcal{L}_{\pm m} & \longrightarrow & N & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & Z & \longrightarrow & \mathcal{L}_1 & \longrightarrow & N & \longrightarrow & 1
\end{array}$$

Note we have used 5.2 to select $\pm m$ so that the restriction of ψ to the center will be the identity. Clearly $\phi^*(a) = \pm ma$. The sufficiency is obvious, and does not require the assumption N is centerless.

6. Semi-direct products

Suppose now that (Z_n, π) is an action of Z_n on π as a group of automorphisms. Let $T: \pi \to \pi$ be the automorphism corresponding to the generator $\sigma(1) \in Z_n$. For each q with $(q, n) = 1$ we introduce a semi-direct product group L_q with multiplication in $\pi \times Z$ given by $(\alpha, s)(\beta, t) = \left(\alpha T^{qs}(\beta), s+t\right)$. There is also the group extension

$$1 \longrightarrow \pi \longrightarrow \pi \circ Z_n \longrightarrow Z_n \longrightarrow 1$$

with the product in $N = \pi \circ Z_n$ given by

$$\left(\alpha, \sigma(s)\right)\left(\beta, \sigma(t)\right) = \left(\alpha T^{s}(\beta), \sigma(s+t)\right)$$

Of course we may also form the groups $\mathcal{L}_m$ for all $(m, n) = 1$.

6.1 Lemma: If $mq \equiv 1 \bmod n$, then

$$\mathcal{L}_m \simeq L_q .$$

Proof: An element in $\mathcal{L}_m$ is a triple $\left(\alpha, \sigma(s), k\right)$ where $\alpha \in \pi$, $\sigma(s) \in Z_n$, $k \in Z$ with $\sigma(k) = m\sigma(s)$; that is, $k - ms \equiv 0 \bmod n$, or $qk \equiv s \bmod n$. The isomorphism is

$$\left(\alpha, \sigma(s), k\right) \longrightarrow (\alpha, k).$$

If $\left(\beta, \sigma(t), k'\right) \in \mathcal{L}_m$ also, then

$$\left(\alpha, \sigma(s), k\right) \cdot \left(\beta, \sigma(t), k'\right) = \left(\alpha T^{s}(\beta), \sigma(s+t), k+k'\right)$$

but $(\alpha, k)(\beta, k') = \left(\alpha T^{qk}(\beta), k+k'\right) = \left(\alpha T^{s}(\beta), k+k'\right)$ which shows the correspondence is a homomorphism. A kernel element in the kernel has the form $\left(e, \sigma(s), 0\right)$ with $\sigma(ms) = 0$ so that $s \equiv 0 \bmod n$ and hence we have a monomorphism. For a given k, $ms \equiv k \bmod n$ has a solution s, thus

$$\mathcal{L}_m \simeq L_q$$

as asserted.

6.2 Lemma: If T has order exactly n in $\mathrm{Out}(\pi)$, then the center of N is trivial if and only if the identity element is the only central element of π which is fixed under T.

Proof: Suppose $\left(\alpha, \sigma(s)\right) \in N$ is a central element, then for all $\beta \in \pi$

$$\big(\alpha,\sigma(s)\big)(\beta,0) = \big(\alpha T^{s}(\beta),\sigma(s)\big) = \big(\beta\alpha,\sigma(s)\big) = (\beta,0)\big(\alpha,\sigma(s)\big) .$$

Thus $T^{s}(\beta) \equiv \alpha^{-1}\beta\alpha$, but T has order n in $\mathrm{Out}(\pi)$, hence $s \equiv 0 \bmod n$, α is central. But $(\alpha,0)\big(e,\sigma(1)\big) = \big(\alpha,\sigma(1)\big) = \big(e,\sigma(1)\big)(\alpha,0) = \big(T\alpha,\sigma(1)\big)$; that is, α is a central element of π fixed under T. Thus we see when (5.5) may apply to this semi-direct product case.

We note in every case that $\pi \subset L_q$ is a normal subgroup. We wish to impose a condition which will guarantee us that π is a characteristic subgroup.

6.3 Lemma: [9 ; 4.1] . *If $I - T_*^q : H_1(\pi;Q) \simeq H_1(\pi;Q)$ then $\pi \subset L_q$ is the kernel of the natural homomorphism*

$$L_q \longrightarrow (L_q/[L_q,L_q]) \otimes_Z Q .$$

Proof: We apply the Lyndon spectral sequence to $0 \longrightarrow \pi \longrightarrow L_q \longrightarrow Z \longrightarrow 0$ for computing $H_1(L_q;Q)$ noting $\chi : Z \longrightarrow L_q$ given by $k \rightarrow (e,k)$ splits the sequence. Then

$$E^2_{1,0} \simeq H_1\big(Z;H_0(\pi;Q)\big) \simeq H_1(Z;Q)$$

$$E^2_{0,1} \simeq H_0\big(Z;H_1(\pi;Q)\big)$$

But Z acts on $H_1(\pi;Q)$ by T_*^q, and since $I - T_*^q : H_1(\pi;Q) \simeq H_1(\pi;Q)$ it follows $E^2_{0,1} = 0$. Thus we have $\chi_* : H_1(Z;Q) \simeq H_1(L_q;Q)$ and $\pi \longrightarrow H_1(\pi;Q) \longrightarrow H_1(L_q;Q)$ is trivial. Any element of L_q can be written $(\alpha,0)\cdot(e,k)$ and $(\alpha,0)$ lies in the kernel of $L_q \longrightarrow H_1(L_q;Q)$ while (e,k) does if and only if $k = 0$, so the lemma follows.

If $H_1(\pi;Q)$ is finite dimensional then $I - T_*^q : H_1(\pi;Q) \simeq H_1(\pi;Q)$ for all q with $(q,n) = 1$ if and only if $0 \in H_1(\pi;Q)$ is the only element fixed under T_*.

6.4 Theorem: *If (Z_n,π) is a cyclic group of automorphisms on a group for which $H_1(\pi;Q)$ is finite dimensional and $I - T_* : H_1(\pi;Q) \simeq H_1(\pi;Q)$ then*

$$L_q \simeq L_1$$

if and only if T^q is conjugate to $T^{\pm 1}$ in $\mathrm{Out}(\pi)$.

Proof: By the above lemma any isomorphism $\Psi : L_q \rightarrow L_1$ will preserve the subgroup π, and hence induce an automorphism on $Z = L_q/\pi = L_1/\pi$. Thus we can write

$$\Psi(\alpha, 0) = (c(\alpha), 0)$$

$$\Psi(e, 1) = (\delta, \pm 1)$$

where c is some automorphism of π and $\delta \in \pi$. However,

$$\Psi((e,1)(\alpha,0)) = (\delta, \pm 1)(c(\alpha), 0) = \Psi(T^q(\alpha), 1)$$

$$= \Psi((T^q(\alpha), 0)\cdot(e,1)) = (c(T^q(\alpha)))(\delta, \pm 1) \quad .$$

Upon multiplying out in L_1 we have

$$\delta T^{\pm 1}(c(\alpha)) = c(T^q(\alpha))\delta$$

or $T^{\pm 1}(c(\alpha)) = \delta^{-1}(cT^q(\alpha))\delta$. Plainly it follows T^q is conjugate by c to $T^{\pm 1}$ in $\mathrm{Out}(\pi)$.

To prove sufficiency we observe by (5.2) that $L_q \simeq L_{-q}$ so it is enough to consider an automorphism $c:\pi \simeq \pi$ and a $\delta \in \pi$ with $Tc(\alpha) = \delta^{-1}(c(T^q(\alpha)))\delta$. There is, with respect to T_* a crossed-homomorphism $\phi: Z \longrightarrow \pi$ with $\phi(1) = \delta$. This is seen by remarking Z is free and extending to a homomorphism $Z \longrightarrow L_1$ the assignment $1 \longrightarrow (\delta, 1)$. The homomorphism has the form $k \longrightarrow (\phi(k), k)$ and $\phi: Z \longrightarrow \pi$ is the required crossed homomorphism. By induction the identity

$$\phi(k)(T^k c(\alpha)) = c(T^{qk}(\alpha))\phi(k)$$

may be verified. Then the isomorphism $\Psi: L_q \simeq L_1$ is $\Psi(\alpha, k) = (c(\alpha)\phi(k), k)$. (We call attention to [9; 4.2] for comparison for the sufficiency.)

6.5 Corollary: *If, under the hypothesis of (6.4), $L_q \simeq L_1$, then in the group of degree preserving ring automorphisms of $H^*(\pi;Z)$, $(T^q)^*$ is conjugate to $(T^{\pm 1})^*$.*

Proof: Since elements of $\mathrm{Out}(\pi)$ induce unique automorphisms of $H^*(\pi; Z)$ the assertion follows.

6.6 An elementary example is obtained by taking $\pi = Z_p$, with p a prime, putting $n = p-1$ so that Z_{p-1} acts on Z_p as the group of automorphisms and $\mathrm{Aut}(Z_p) = \mathrm{Out}(Z_p) = Z_{p-1}$, an abelian group. In this case, if $(q, p-1) = 1$, then $L_q \simeq L_1$ if and only if $q \equiv \pm 1 \mod (p-1)$. Observe that $H_1(Z_p; Q) = 0$. One could take, for example $(q, p-1) = (3, 10)$. We may present the 2 groups:

$$Z \times (Z_{11} \circ Z) = Z \times L_1 = \{x, y, z \mid x^{11} = 1,\ yxy^{-1} = x^2,\ yz = zy,\ xz = zx\}$$

$$Z \times (Z_{11} \circ Z) = Z \times L_3 = \{\bar{x}, \bar{y}, \bar{z} \mid \bar{x}^{11} = 1,\ \bar{y}\bar{x}\bar{y}^{-1} = \bar{x}^8,\ \bar{y}\bar{z} = \bar{z}\bar{y},\ \bar{x}\bar{z} = \bar{z}\bar{x}\} .$$

Define $h: L_1 \longrightarrow L_3$, $k: L_3 \longrightarrow L_1$, by

$$h(x) = \bar{x},\ \ h(y) = \bar{y}^7 \bar{z}^{10},\ \ h(z) = \bar{y}^2 \bar{z}^3$$

$$k(\bar{x}) = x,\ \ k(\bar{y}) = y^3 z^{-10},\ \ k(\bar{z}) = y^{-2} z^7 \quad .$$

It can be easily checked that $h \circ k$ = identity and $k \circ h$ = identity.

7. Charlap's example

The idea for this entire study was suggested by an example in a paper of L. Charlap, [2]. Let $\pi = (Z)^k$ be a free abelian group of rank k. Suppose for some prime p there is an automorphism T on π, with period p, which leaves no element other than 0 fixed. We note immediately that $I - T_*: H_1(\pi; Q) \simeq H_1(\pi; Q)$. In addition, with $N = \pi \circ Z_p$ via T, we can also apply (6.2) to see that N is centerless, hence by (5.4) and (6.4) we have

7.1 Theorem: For a Charlap example the following are equivalent, with $(q, p) = 1$:

1) $L_q \simeq L_1$
2) there is an automromphism $\phi: N \simeq N$ with $\phi^*(a) = \pm ma$, $mq \equiv 1 \pmod p$
3) T^q is conjugate to $T^{\pm 1}$ in $GL(k, Z)$.

With the aid of Reiner's theorem [15] it is possible to answer the conjugacy problem by means of an invariant which lies in the group of ideal classes for the cyclotomic number field $Q(\lambda)$ obtained by adjoining the p^{th} roots of unity to Q. Denote this abelian group by C and by $a \in C$ the equivalence class of a fractional ideal. The group is written multiplicatively with the equivalence class of the algebraic integers, $Z(\lambda)$, being the identity. Now the Galois group acts on C as a group of automorphisms $a \longrightarrow a^q$ where $1 \le q \le p-1$ is regarded as the obvious element of the Galois group.

We assumed T left no element fixed in $\pi = (Z)^k$ other than 0 so by Reiner's theorem π, as a $Z(Z_p)$-module, is isomorphic to a direct sum $Z(\lambda) \oplus \ldots \oplus Z(\lambda) \oplus A$, where $A \subset Q(\lambda)$ is a fractional ideal. Note immediately $k \equiv 0 \bmod (p-1)$. The equivalence class of A in C is the invariant. Thus T^q is conjugate to $T^{\pm 1}$ in $GL(k, Z)$ if and only if $A^q = A^{\pm 1}$ in C. This simply tells us T^q and $T^{\pm 1}$ define $Z(Z_p)$-module structures on π, which in view of Reiner's result can be isomorphic if and only if $A^q = A^{\pm 1} \in C$.

The group C which is finite is extremely difficult to deal with effectively. However, according to Kümmer, for irregular primes (the least of which is 37) there is a cyclic subgroup in C of order p. By taking $k = p-1$ and $\pi = A$, a generator of this subgroup, we see $L_q \simeq L_1$ if and only if $q \equiv \pm 1 \pmod p$.

The groups L_q in this case are all Bieberbach groups; that is, fundamental groups of flat, compact, Riemannian manifolds.

The examples of (6.6) and (7.1) may be readily geometrically realized as in § 4. One needs in (6.6) to construct a space with fundamental group Z_p on which Z_{p-1} operates so that the induced automorphism of the fundamental group is Aut Z_p. It would be interesting if this could be done for some lens space. However, one can take any lens space with fundamental group Z_p and take the (p-1)-fold Cartesian product of the universal cover. This is a (p-1)-fold product of the same sphere. On this product space the group $Z_p \circ Z_{p-1}$ smoothly acts with Z_p acting freely. The action of Z_{p-1} induces the action of the automorphisms of Z_p on the fundamental group of the closed quotient manifold Y of the free Z_p-action. See [9 , 2.5] for the construction and the details. We form

$$X(1) = S^1 \times_{Z_{p-1}^{(1)}} Y \qquad \text{and} \qquad X(q) = S^1 \times_{Z_{p-1}^{(2)}} Y .$$

The fundamental groups are: $\pi_1(X(1)) \cong Z \circ Z_p = L_1$ and $\pi_1(X(q)) \approx Z \circ Z_p = L_q$. By § 4 $S^1 \times X(1) = S^1 \times X(q)$. But, by §6.6 $\pi_1(X(1))$ is not isomorphic to $\pi_1(X(q))$. This rather crude construction yields closed manifolds of dimension at least 31.

Charlap, in his paper [2], realized his examples as closed flat manifolds. This is given by choosing in $GL(p-1, Z)$ the necessary faithful representation of Z_p. The group Z_p operates on T^{p-1} and $X(q) = S^1 \times_{Z_p^{(2)}} T^{p-1}$. Thus the closed flat manifolds X(q) appear to have dimension at least 37.

In the next sections we shall describe how we may find, by exploiting the techniques of § 5 and § 6, closed smooth manifold examples in much lower dimensions.

8. Charlap actions and the Atiyah-Bott formula

Let us consider a cyclic group (Z_n, Y) of orientation preserving diffeomorphisms on a closed orientable aspherical manifold. (A manifold is aspherical if it is a $K(\pi, 1)$.) If in addition (Z_n, Y) has at least one fixed point and $H^1(Y/Z_n; Z) = 0$ we shall say that (Z_n, Y) is a Charlap action. For each integer q, $(q, n) = 1$, there is associated to (Z_n, Y) a closed aspherical manifold X(q), fibered over S^1 with fiber Y and structure group Z_n. Furthermore $S^1 \times X(q)$ is diffeomorphic to $S^1 \times X(1)$. Let $(T_*, \pi_1(Y, y_0))$ be the automorphism corresponding to the generator $\lambda = \exp(2\pi i/n)$ in Z_n.

8.1 Theorem: If (Z_n, Y) is a Charlap action then $X(q)$ has the homotopy type of $X(1)$ if and only if T_*^q is conjugate in $\mathrm{Out}(\pi)$ to $T_*^{\pm 1}$.

Since we are dealing with aspherical spaces, homotopy equivalence is equivalent to an isomorphism between the fundamental groups. The theorem then is immediately implied by (6.4).

We actually need not consider the case T^q conjugate in $\mathrm{Out}(\pi)$ to T^{-1} as separate. Simply note that if in $\mathrm{Out}(\pi_1(Y, y_0))$

$$c^{-1} \circ T^{-1} \circ c = T^q$$

then $c^{-1} \circ T \circ c = T^{-q}$ in $\mathrm{Out}(\pi_1(Y, y_0))$. However, $X(q)$ is homeomorphic to $X(-q)$. In fact the homeomorphism is $((t, y))_q \longrightarrow ((t^{-1}, y))_{-q}$.

Thus, for a Charlap action we should like to characterize those integers q with

(i) $(q, n) = 1$

(ii) T_*^q is conjugate to T_* in $\mathrm{Out}(\pi_1(Y, y_0))$.

We note trivially that for any such q, the translate $q+n$ is also in this set. We are only able to solve this problem completely in a few cases. However, we can give, in terms of the Atiyah-Bott fixed point formula, a necessary condition that T_*^q be conjugate to T_* in $\mathrm{Out}(\pi_1(Y, y_0))$. We recall that if $\dim Y \equiv 0 \pmod 2$ then to the diffeomorphisms (T, Y) and (T^q, Y) there are associated invariants Index (T, Y) and $\mathrm{Ind}(T^q, Y)$ in $Z(\lambda) \subset C$. These are defined in terms of the induced automorphisms on $H^*(Y; R)$. The Atiyah-Bott formula computes these algebraic integers in terms of the fixed point set of (T, Y). We wish to show

8.2 Theorem: If (Z_n, Y) is a Charlap action and if q is an integer for which $(q, n) = 1$ and T_*^q is conjugate in $\mathrm{Out}(\pi_1(Y, y_0))$ to T_* then

$$\mathrm{Ind}(T^q, Y) = \pm \mathrm{Ind}(T, Y).$$

The proof rests on the fact that Y is a $K(\pi, 1)$ so that $H^*(\pi; R) \simeq H^*(Y; R)$ is determined entirely by the fundamental group. We wish to approach this matter in a general manner.

We regard the reals, R as a trivial $Z(\pi)$-module.

8.3 Definition: The group π is an orientable real Poincaré group if and only if

(i) $H_*(\pi; R)$ is finite dimensional

(ii) there is an integer $n \geq 0$ and a $\sigma \in H_n(\pi; R)$ such that for every integer j the linear transformation given by $c^j \longrightarrow c^j \cap \sigma$ is an isomorphism $H^j(\pi; R) \simeq H_{n-j}(\pi; R)$.

In particular $H^0(\pi; R) \simeq H_n(\pi; R) \cong R$. The nonzero elements of $H_n(\pi; R)$ are divided into two equivalence classes $\mathfrak{o}$ and $-\mathfrak{o}$, called the orientations, by the relation $\sigma_0 \sim \sigma_1$ if and only if $r\sigma_0 = \sigma_1$ for some $r > 0$. It is also a corollary of duality that $H^j(\pi; R) = H_j(\pi; R) = 0$ if $j > n$, thus we write $\dim_R \pi = n$.

8.4 Definition: If $(\pi, \mathfrak{o})$ and $(\pi', \mathfrak{o}')$ are oriented real Poincaré groups with $\dim_R \pi = \dim_R \pi' = n$ then a homomorphism $\phi: \pi \longrightarrow \pi'$ is orientation preserving if and only if

(i) $\phi_*: H_n(\pi; R) \simeq H_n(\pi'; R)$

(ii) for $\sigma \in \mathfrak{o}$ then $\phi_*(\sigma) \in \mathfrak{o}'$.

For any oriented real Poincaré group we may thus identify the subgroup $\mathrm{Saut}(\pi) \subset \mathrm{Aut}(\pi)$ of orientation preserving automorphisms of π. This is a normal subgroup of index at most 2 which does not depend on the choice of orientations.

We shall regard $\mathrm{Aut}(\pi)$ as acting on the right on π so that it will act, by induced automorphisms, from the right on $H_*(\pi; R)$ and from the left on $H^*(\pi; R)$. Since every inner-automorphism induces the identity on both $H_*(\pi; R)$ and $H^*(\pi; R)$ we see that $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ acts on both homology and cohomology. By the same token $\mathrm{Inn}(\pi) \subset \mathrm{Saut}(\pi)$, so we obtain a subgroup $\mathrm{Sout}(\pi) = \mathrm{Saut}(\pi)/\mathrm{Inn}(\pi) \subset \mathrm{Out}(\pi)$ of orientation preserving outer automorphisms.

If F is a finite group and if $(\pi, \mathfrak{o})$ is an oriented real Poincaré group with $\dim_R \pi \equiv 0 \pmod 2$ then to each homomorphism $\psi: F \rightarrow \mathrm{Out}(\pi)$ we shall assign $\mathrm{Ind}_{\mathfrak{o}}(\psi) \in R_c(F)$, the Grothendieck ring of finite dimensional complex representation classes of F. Further, $\mathrm{Ind}(\psi)$ depends up to sign only on the conjugacy class of ψ in $\mathrm{Out}(\pi)$. We shall only give a brief description of $\mathrm{Ind}_{\mathfrak{o}}(\psi)$ as it is a standard definition.

We denote by x^* the automorphism of $H^{n/2}(\pi; R)$ induced by $x \in F$ via $\psi(x)$. Select a representative $\sigma \in \mathfrak{o}$ and note that since $\psi: F \rightarrow \mathrm{Sout}(\pi)$, $x_*(\sigma) = r\sigma$, for some $r > 0$. But F is a finite group so that $x_*(\sigma) = \sigma$ for all $x \in F$. On $H^{n/2}(\pi; R)$ we introduce a real bi-linear non-singular inner product

$$(v, w) = \langle v \cup w, \sigma \rangle = \epsilon_* (v \cup w) \cap \sigma \in R$$

Then $(v,w) = (-1)^{n/2}(w,v)$. Furthermore

$$\left(x^*(v), x^*(w)\right) = \langle x^*(v) \cup x^*(w), \sigma\rangle$$
$$= \langle x^*(v \cup w), \sigma\rangle$$
$$= \langle v \cup w, x_*(\sigma)\rangle = \langle v \cup w, \sigma\rangle = (v,w).$$

There is a real linear operator $D: H^{n/2}(\pi; R) \longrightarrow H^{n/2}(\pi; R)$ such that

(i) $D \circ x^* = x^* \circ D$, all $x \in F$

(ii) $D^2 = (-1)^{n/2} I$

(iii) $(v, Dw) = (-1)^{n/2}(Dv, w)$

(iv) $(v, Dv) > 0$ if $v \neq 0$.

If $n/2$ is odd, then $D = J$ is a complex structure on $H^{n/2}(\pi; R)$ and $\left(F, H^{n/2}(\pi; R), J\right)$ is a complex representation, on a finite dimensional space, and we put

$$\mathrm{Ind}_{\sigma}(\Psi) = \left(F, H^{n/2}(\pi; R), J\right) - \left(F, H^{n/2}(\pi; R), -J\right)$$

in $R_c(F)$ if $n/2$ is odd. Thus in this case $\mathrm{Ind}_{\sigma}(\Psi)$ is the difference between a complex representation and its conjugate. If $n/2$ is even then $D^2 = I$ and we split $H^{n/2}(\pi; R)$ into F-invariant subspaces

$$V_+ = \{v \mid v = Dv\}$$
$$V_- = \{v \mid -v = Dv\}$$

We let $\mathrm{Ind}_{\sigma}(\Psi) = (F, V_+ \otimes_R C) - (F, V_- \otimes_R C)$ in $R_c(F)$. If $r\sigma$, $r > 0$, replaces σ then (v,w) is replaced by $r(v,w)$. But then the operator D still satisfies $r(v, Dv) > 0$, $v \neq 0$, hence $\mathrm{Ind}_{\sigma}(\Psi)$ depends only on the orientation σ, not on the representative. Now if σ is replaced by $-\sigma$ then we replace (v,w) by $-(v,w)$ and we must replace D by $-D$ to have $-(v, -Dv) > 0$ if $v \neq 0$. Hence $\mathrm{Ind}_{-\sigma}(\Psi) = -\mathrm{Ind}_{\sigma}(\Psi)$.

8.5 Lemma: *If $\phi: \pi \simeq \pi$ is an automorphism and if $\Phi: F \to \mathrm{Sout}(\pi)$ is given by*

$$\Phi(x) = \phi \circ \Psi(x) \circ \phi^{-1}$$

then $\mathrm{Ind}_{\sigma}(\Phi) = \pm \mathrm{Ind}_{\sigma}(\Psi)$ according to whether or not ϕ is orientation preserving.

We write $X^* = \phi^* \circ x^* \circ (\phi^{-1})^*$, which is the automorphism of $H^*(\pi; R)$. We can write $r\sigma = \phi_*(\sigma)$ for a unique $r \neq 0$. Then $\left(\phi^*(v), \phi^*(w)\right) = <\phi^*(v \cup w), \sigma> = <v \cup w, \phi_*(\sigma)> = r(v,w)$. As a corollary

$$\left(v, \phi^*(w)\right) = \left(\phi^*(\phi^{-1})^* v, \phi^*(w)\right) = r\left((\phi^{-1})^* v, w\right)$$

$$\left(v, (\phi^{-1})^* w\right) = (-1)^{n/2}\left((\phi^{-1})^* w, v\right) = (-1)^{n/2}(1/r)\left(w, \phi^*(v)\right)$$

$$= 1/r\left(\phi^*(v), w\right) .$$

We now replace the operator D by $D' = \phi^* \circ D \circ (\phi^{-1})^*$. We see trivially that $X^* \circ D' = D' \circ X^*$ and $(D')^2 = (-1)^{n/2} I$. Now

$$\left(v, \phi^* \circ D \circ (\phi^{-1})^*(w)\right) = r\left((\phi^{-1})^* v, D \circ (\phi^{-1})^*(w)\right)$$

$$= \pm r\left(D \circ (\phi^{-1})^* v, (\phi^{-1})^*(w)\right) = \pm\left(\phi^* \circ D \circ (\phi^{-1})^* v, w\right) .$$

Finally $(v, D'v) = r\left((\phi^{-1})^*(v), D(\phi^{-1})^* v\right)$. Thus if $r > 0$ $(v, D'v) > 0$, but if $r < 0$, then $(v, -D'v) > 0$. We apply the definition of $\mathrm{Ind}_\sigma(\Phi)$ using $\pm D'$, $X^* \circ \phi^* = \phi^* \circ x^*$, we see immediately that

$$\mathrm{Ind}_\sigma(\Phi) = \pm \mathrm{Ind}_\sigma(\Psi) .$$

Now suppose $T \in \mathrm{Sout}(\pi)$ is an element of order $n > 1$. There is a unique homomorphism

$$\Psi : Z_n \longrightarrow \mathrm{Sout}(\pi)$$

with $\Psi(\lambda) = T$. We then set

$$\mathrm{Ind}_\sigma(T) = \mathrm{Tr}\left(\mathrm{Ind}_\sigma(\Psi)\right)$$

where $\mathrm{Tr}: R_c(Z_n) \longrightarrow Z(\lambda)$ is the trace homomorphism whose kernel is the ideal generated by the regular representation. If T^q is conjugate to T in $\mathrm{Out}(\pi)$ we know $\mathrm{Ind}_\sigma(T^q) = \pm \mathrm{Ind}_\sigma(T)$.

Now $\mathrm{Ind}_\sigma(T)$ is computed as follows. Consider first $\dim_R \pi = 4k$, and let T^* be the automorphism on $H^{2k}(\pi; C)$ induced by T. Recall $H^{2k}(\pi; C)$ was split into a sum of T^*-invariant subspaces $V^+ \otimes_R C$ and $V^- \otimes_R C$. For each i, $0 \leq i < n$, let m_i^+ be the multiplicity of the eigenvalue λ^i in $V^+ \otimes_R C$ and m_i^- is the multiplicity in $V^- \otimes_R C$. Let $m_i = m_i^+ - m_i^-$, then

$$\operatorname{Ind}_{\sigma}(T) = \sum_{i=0}^{n-1} m_i \lambda^i$$

$$= \sum_{i=1}^{n-1} (m_i - m_0)\lambda^i .$$

Denote by $[qi]$ the integer

$$0 \leq [qi] < n$$

$$[qi] \equiv qi \bmod n .$$

8.6 Lemma: If $n = p$, a prime, and if $\dim_R(\pi) = 0 \bmod 4$, then, for $(q,n) = 1$

$$\operatorname{Ind}_{\sigma}(T) = \operatorname{Ind}_{\sigma}(T^q)$$

if and only if $m_i = m_{[qi]}$, $0 < i < p$,

$$\operatorname{Ind}_{\sigma}(T) = -\operatorname{Ind}_{\sigma}(T^q)$$

if and only if $m_{[qi]} = 2m_0 - m_i$, $0 < i < p$.

Suppose $M_i = M_i^+ - M_i^-$ where M_i^+ is the multiplicity of the eigenvalue λ^i on $V^+ \otimes_R C$ for $(T^*)^q$, and similarly for M_i^-. Trivially we see that $M_{[qi]} = m_i$, $0 \leq i < n$. In particular $M_0 = m_0$. The lemma now follows since $\lambda, \ldots, \lambda^{p-1}$ is an additive basis for $Z(\lambda)$.

If $\dim_R(\pi) = 4k+2$ we recall that $\left(T^*, H^{2k+1}(\pi; R), J\right)$ is complex linear. Let m_i, $0 \leq i < n$ be the multiplicity of λ^i in this representation. Then in the conjugate representation the multiplicity of λ^i will be m_{n-i} of course. Thus in this case

$$\operatorname{Ind}(T) = \sum_{i=1}^{n-1} (m_i - m_{n-i})\lambda^i .$$

8.7 Lemma: If $n = p$, a prime, and if $\dim_R(\pi) \equiv 2 \bmod 4$ then, for $(q,n) = 1$,

$$\operatorname{Ind}_{\sigma}(T^q) = \pm \operatorname{Ind}_{\sigma}(T)$$

if and only if

$$m_i - m_{p-i} = \pm(m_{[qi]} - m_{p-[qi]}), \qquad 0 < i < p.$$

The proof is like the above.

Every remark applies immediately to Charlap actions (Z_n, Y), whence Theorem 8.2 now follows. However, in this case, the Atiyah-Bott fixed point formula can be applied to compute $\mathrm{Ind}(T)$ in terms of fixed point data.

As an illustrative example, fix an odd prime p and introduce a curve $S \subset CP(2)$ by

$$S = \left\{ \left[z_1, z_2, z_3\right] \middle| z_1^p + z_2^p + z_3^p = 0 \right\} .$$

which is a Riemann surface of genus $(p-1)(p-2)/2 > 0$, and hence is a closed aspherical manifold. With $\lambda = \exp 2\pi i/p$ define (T, S) by $T\left[z_1, z_2, z_3\right] = \left[\lambda z_1, z_2, z_3\right]$. Since p is odd there are exactly p fixed points $\left\{\left[0, -1, \lambda^i\right]\right\}_{i=0}^{p-1}$. The map $S \longrightarrow CP(1)$ given by $\left[z_1, z_2, z_3\right] \longrightarrow \left[z_2, z_3\right]$ coincides with the quotient map $S \longrightarrow S/T$. Since $H^1\left(CP(1); Z\right) = 0$ we know (Z_p, S) is a Charlap action.

Now there is also on S a complex analytic, T-equivariant, periodic map

$$f\left[z_1, z_2, z_3\right] = \left[z_1, z_2, \lambda z_3\right]$$

which cyclicly permutes the fixed points of (T, S). Thus at every fixed point we will see the same complex p-dimensional representation of Z_p appears in the tangent line. It happens to be multiplication by λ, although this is not essential for what follows.

According to the Atiyah-Bott fixed point formula, then,

$$-p\big((1+\lambda)/(1-\lambda)\big) = \mathrm{Ind}(T_*) \quad .$$

If T is replaced by T^q, then the local representation at each fixed point becomes multiplication by λ^q so

$$-p\left((1+\lambda^q)/(1-\lambda^q)\right) = \pm \mathrm{Ind}(T_*^q)$$

where the sign of the right is $+$ if $1 \leq q \leq p-1/2$, and $-$ if $p-1/2 < q \leq p-1$. In any case, we are concerned with finding the q, $1 < q < p-1$, which satisfy

$$\big((1+\lambda)/(1-\lambda)\big) = \pm\big((1+\lambda^q)/(1-\lambda^q)\big)$$

or equivalently

$$(1+\lambda)(1+\dots+\lambda^{q-1}) = \pm(1+\lambda^q)$$

This is impossible if $1 < q < p-1$, for it would imply either

$$\lambda+\ldots+\lambda^{q-1}=0$$

or

$$1+\lambda+\ldots+\lambda^{q}=0\ .$$

<u>Thus for</u> $1<q<p-1$ <u>the closed</u> 3-<u>manifolds</u> $X(q)$ <u>and</u> $X(1)$ <u>have distinct homotopy types</u>.

9. Circle actions on 3-manifolds

In the last section we gave 3-dimensional examples of Charlap actions with $\pi_1(X(1)) \neq \pi_1(X(q))$. The non-isomorphism was detected by the Index of T from the Atiyah-Bott formula. Inasmuch as all actions of the circle on 3-manifolds are known, we shall, with the aid of Theorem 5.5, be able, in §10, to completely solve the homeomorphism problem for $X(1)$ and $X(q)$ in the 3-dimensional case. The methods which really appear to be special to 3-manifolds do admit a bordism generalization to higher dimensions. We shall pursue this generalization in §12.

For simplicity of exposition we shall consider only orientable 3-manifolds. In [14] the topological actions of the circle on 3-manifolds were classified and shown to be equivalent to certain standard ones. We shall now describe the standard actions for closed oriented 3-manifolds with no fixed points.

We begin with a closed oriented 2-manifold B and form the product $S^1\times B$. In B we select a finite set $\{d_0, d_1, \ldots, d_n\} = E$ and choose a smooth closed disjoint disk D_j centered at each d_j. We delete the interiors of $S^1\times D_j$ from $S^1\times B$. We choose for each $j>0$, a relatively prime pair (α_j,β_j) of integers so that $0<\beta_j<\alpha_j$. We form an action of the circle on a solid torus V_j by

$$(\rho e^{i\theta}, e^{i\psi}) \longrightarrow (z^{\nu_j}\rho e^{i\theta}, z^{\alpha_j}e^{i\psi})$$

where $\nu_j\beta_j \equiv 1 \bmod \alpha_j$. On V_0 we define the action

$$(\rho e^{i\theta}, e^{i\psi}) \longrightarrow (z^{b}\rho e^{i\theta}, z e^{i\psi})\ .$$

We choose a global cross section

$$\chi : B - \bigcup_{j\geq 0} \mathring{D}_j \longrightarrow S^1\times(B-\bigcup \mathring{D}_j)$$

and orient $S^1 \times B$ by means of this section and the standard orientation of S^1. We now attach each V_j to the deleted solid tori by an equivariant orientation reversing homeomorphism along the boundaries so that the resulting 3-manifold M^3 will be oriented and have a natural circle action. There is a natural map by collapsing orbits to points back onto B, where the inverse image of $(B - E) \cup d_0$ are all principal orbits and where any point on the inverse image of d_j, $j > 0$ would have slice representation $Z_{\alpha_j} \times D_j \rightarrow D_j$ given by

$$\rho e^{i\theta} \longrightarrow \rho e^{i\theta} \exp\left(\frac{2\pi i \nu_j}{\alpha_j}\right) .$$

This oriented closed 3-manifold M with its S^1-action is called a standard action. Associated with (S^1, M) is an orientation ϵ, the genus g of the orbit space, the set of oriented Seifert invariants $\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$, and the integer b.

For each arbitrary topological action without fixed points of the circle S^1 on a closed oriented 3-manifold X, one may also find a system of invariants. The orbit space is an oriented closed 2-manifold of genus g. There are a finite number n of orbits where the stability groups are $Z_{\alpha_1}, \ldots, Z_{\alpha_n}$ and the slice representation is topologically equivalent to

$$\rho e^{i\theta} \longrightarrow \rho e^{i\theta} \exp\left(\frac{2\pi i \nu_j}{\alpha_j}\right) .$$

Finally, if we delete invariant tubular neighborhoods of these singular orbits and take a cross-section, on the boundary, to the orbit map and try to extend this section to the rest of the deleted X/S^1, we obtain an integral obstruction cohomology class b. The cross-section extends everywhere except for one point and this integer b measures the chance of extending the cross-section across the final point or disk neighborhood of this point. Thus we see that (S^1, X) has a completely analogous set of invariants as a standard action (S^1, M). The main theorem of [14] implies

9.1 Lemma: For the action (S^1, X),

$$\{\epsilon; g; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$$

is a complete set of invariants.

This means that (S^1, X) is equivariantly homeomorphic to the standard (S^1, M) with the same set of oriented invariants by means of an orientation preserving homeomorphism. If we allow orientation reversing homeomorphisms and/or automorphisms $S^1 \longrightarrow S^1$ sending $z \longrightarrow z^{-1}$ then the set of invariants

$$\left\{\epsilon; g; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\right\}$$

may be transformed into the oppositely oriented set

$$\left\{-\epsilon; g; -b-n; (\alpha_1, \alpha_1-\beta_1), \ldots, (\alpha_n, \alpha_n-\beta_n)\right\}$$

Thus (S^1, X) is equivariantly homeomorphic, allowing the inversion automorphism of S^1, to $((S^1, X')$ if and only if the set

$$\left\{\epsilon; g; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\right\}$$

is equal to either

$$\left\{\epsilon'; g'; b'; (\alpha'_1, \beta'_1), \ldots, (\alpha'_n, \beta'_n)\right\} \quad , \qquad \text{or}$$

$$\left\{-\epsilon', g', -b'-n; (\alpha'_1, \alpha'_1-\beta'_1), \ldots, (\alpha'_n, \alpha'_n-\beta'_n)\right\}$$

The lemma is the equivariant classification of circle actions on 3-manifolds. It remains to describe the topological classification. That is, to decide when two distinct standard actions are homeomorphic. It turns out that only lens spaces, including the 3-sphere and $S^2 \times S^1$, admit more than one action of the circle. Furthermore, the set of invariants, and hence the action and the topological type is completely determined by the fundamental group whenever X is not a lens space. This is proved in [12], [13], and [17]. Since the result is essential for what we shall do here we offer another proof in the spirit of [6] and [8] which generalizes directly to the case of toral T^k-actions on closed $(k+2)$-manifolds, and indirectly to many other interesting situations. Furthermore, by a combination of our technique and of [13], [14], and [17] we may deduce some interesting information about Fuchsian and crystallographic planar groups which do not have compact quotient spaces. We assume familiarity with [6 ;§8] and [8 ; §12]. To eliminate the lens space cases as well as those of finite fundamental group, each of which needs special arguments, we assume that when

$$g = 0, \quad n \geq 3, \quad \text{and if} \quad n = 3, \quad \text{then,}$$

(9.2)
$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \leq 1 .$$

The genus 0-cases eliminated coincide precisely with the actions on lens spaces or those connected oriented 3-manifolds with finite fundamental group which admit fixed point free effective circle action.

It is shown [8 ;§ 12] that in each of the remaining cases, the action (S^1, X) is injective. Furthermore, the splitting action is

$$\left(S^1, X_{\operatorname{im}(f^x_*)}, \pi_1(X,x)/\operatorname{im}(f^x_*)\right) = (S^1, S^1 \times R^2, N) .$$

The projection $(S^1, S^1 \times R^2, N) \xrightarrow{/S^1} (R^2, N)$ induces a properly discontinuous topological action of N on the plane with compact quotient $R^2/N = X/S^1$. Any such action is known to be topologically equivalent to a planar group, that is, an orientation preserving crystallographic or Fuchsian group.

9.3 Theorem: Let (S^1, X) and (S^1, X^1) be injective actions on closed oriented 3-manifolds satisfying 9.2. Then the following are equivalent.

(i) (S^1, X) and (S^1, X') are equivariantly homeomorphic allowing an automorphism of S^1,

(ii) (S^1, X) and (S^1, X') are homeomorphic,

(iii) $\left\{\epsilon; g; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\right\}$ equals

$\left\{\epsilon'; g'; b'; (\alpha'_1, \beta'_1), \ldots, (\alpha'_n, \beta'_n)\right\}$ or

$\left\{-\epsilon'; g'; -b'-n; (\alpha'_1, \alpha'_1-\beta'_1), \ldots, (\alpha'_n, \alpha'_n-\beta'_n)\right\}$,

(iv) $\pi_1(X,x)$ is isomorphic to $\pi_1(X',x')$.

Proof. Obviously, (iii) $\Longleftrightarrow$ (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv). Therefore we prove (iv) $\Longrightarrow$ (i). Since the action is injective, the splitting action $(S^1, S^1 \times R^2, N)$ is represented by a Bieberbach class $a \in A \subset H^2(N; Z)$. This also represents the extension

$$0 \longrightarrow \operatorname{im}(f^x_*) \longrightarrow \pi_1(X,x) \longrightarrow N \longrightarrow 1 .$$

The action (R^2, N) as mentioned earlier is topologically equivalent to an orientation preserving "planar" group with compact quotient. Except for $g = 1$, $n = 0$, where N is isomorphic to $Z \oplus Z$, N is centerless. In the special case (S^1, X) is a principal circle bundle over the torus and one can easily check (iv) $\Longrightarrow$ (i). So we assume also that if $g = 1$, $n > 0$.

We need this following generalization of Nielsen's theorem due to Zieschang [19] and Macbeath [11].

9.4 Lemma: Let $(R^2, N, \cdot)$ and $(R^2, N', *)$ be two effective properly discontinuous actions on the Euclidean plane without reflections and with compact quotient spaces. If $\Phi: N \longrightarrow N'$ is an isomorphism, then there exists a homeomorphism $\Psi: R^2 \to R^2$ so that $\Psi(w \cdot \alpha) = \Psi(w) * \Phi(\alpha)$.

Suppose there exists an isomorphism

$$h:\pi_1(X,x) \longrightarrow \pi_1(X',x') .$$

Then as N and N' are centerless, $\mathrm{im}(f_*^X)$ and $\mathrm{im}(f_*^{X'})$ are the centers and hence characteristic subgroups. Thus h induces an isomorphism $\Phi: N \longrightarrow N'$. Thus, we may as well assume that (S^1, X) and (S^1, X') arise from the same planar action (R^2, N) but of course represented by possibly different elements a, and $a' \in H^2(N; Z)$.

For arbitrary injective toral actions (T^k, X) and (T^k, X') represented by Bieberbach classes a, $a' \in H^2(N; Z^k)$ from the same properly discontinuous actions (W, N) on the simply connected space W we have shown in [6 ; 8.6]

9.5 Lemma: (T^k, X) and (T^k, X') are equivariantly homeomorphic if and only if there exists an automorphism $\Phi: N \longrightarrow N$ and a homeomorphism $\psi: W \longrightarrow W$ so that

$$\psi(w\alpha) = \psi(w)\Phi(\alpha), \qquad w \in W, \ \alpha \in N ,$$

and

$$\Phi^*(a') = a .$$

We remark that if we wish to have homeomorphisms that allow topologically linear automorphisms of the orbits then we must consider equivariant homeomorphisms module automorphisms of T^k.

We apply this now directly when $k = 1$, $W = R^2$ and N is a centerless planar group with compact quotient. The isomorphism h induced the isomorphism Φ, which implies the existence of a ψ with $\Phi^*(a') = a$. This concludes the proof except for the possibility the h restricted to $\mathrm{im}(f_*^X)$ sent the generator to the negative of the generator of $\mathrm{im}(f_*^{X'})$. In this case, oppositely orient the circle without changing the orientation of X'. This sends the invariants (α_j', β_j') into $(\alpha_j', \alpha_j' - \beta_j')$ and b' into $-b'-n$.

If we alter the (iv)-th statement the theorem is still correct for compact orientable 3-manifolds. This is proved in [13] and [17]. It is possible however to establish the result for any (S^1, X), where $\pi_1(X)$ is finitely generated (with the usual exceptions). This will enable us to give some interesting examples in the case of open 3-manifolds. For convenience, we also assume that the boundary of X is always compact.

We first describe the necessary standard examples. We begin again with an oriented closed surface of genus g. We delete the interiors of $s \geq 0$ closed disks F_k, and h points, $y_1, \ldots, y_h$. In the interior of what is left we mark the points $E = (d_1, \ldots, d_n)$ and choose

disjoint disks D_j as before. We call our deleted and marked oriented 2-manifold of genus g, with s boundary components and h holes and n "singular orbits" also by B. We form the product $S^1 \times B$ exactly as before and sew in equivariantly

$$(S^1, S^1 \times_{Z_{\alpha_i}} D)$$

in the deleted $S^1 \times D_j$. We assume that $s+h>0$ so that (S^1,X) is not closed.

The classification theorem of [14] is still valid and it states that every effective (S^1,X) without fixed points on an orientable 3-manifold with compact boundary and finitely generated fundamental group is equivariantly homeomorphic to a standard example. *In this case the*

$$\left\{\epsilon; (g,s,h); (\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\right\}$$

is a complete set of invariants. The integer b does not appear.

In the non-closed case all actions are injective and N is easily seen to be $Z*\ldots*Z*Z_{\alpha_1}*\ldots*Z_{\alpha_n}$, where $*$ denotes the free product. There are $2g+s+h-1$ free Z-factors. Thus, the generalized Nielsen theorem obviously does not extend to the case of planar groups with non-compact quotient space. In order to ensure that N is an infinite centerless group we assume that

(9.6) *If* $g=0$, *then* $n+s+h>2$.

Thus, N operates effectively and properly discontinuously on a simply connected 2-manifold W. It preserves orientation, and $W=R^2$ if and only if $s=0$. Unfortunately, for a fixed N and a Bieberbach class in $H^2(N;Z)$, all the (S^1,X) have isomorphic fundamental groups.

> 9.7. *Lemma: If* (S^1,X) *and* (S^1,X') *are not closed, oriented and satisfy* 9.6, *then they are equivariantly homeomorphic if and only if they are homeomorphic*.

As mentioned earlier, this was proved in the compact case (that is, when $h=0$) in [13] and [17]. To prove this in the general case we compactify X to $\overline{X}$ by compactifying B to $\overline{B}$ by the addition of a circle boundary K_m for each missing y_m in $\{y_1,\ldots,y_h\}$. Thus B is embedded in $\overline{B}$ where $\overline{B}$ is compact and has $h+s$ boundary components. The action (S^1,X) is embedded in $(S^1,\overline{X})$. Let $H:X\longrightarrow X'$ be the given homeomorphism. Since $\overline{X}$ is collared at the boundary of $\overline{X}$, we may write $\overline{X}_1\subset X\subset \overline{X}_1\cup(\partial\overline{X}\times[0,1])=\overline{X}$. If we restrict H to $\overline{X}_1$, then $H(\overline{X}_1)\subset X'\subset\overline{X'}$ and there is an h-cobordism between the components of the toral boundaries. By Waldhausen's theorem [18] this is a product and thus H may be extended to a

homeomorphism $\overline{X} \longrightarrow \overline{X'}$. Therefore the extended H is homotopic to an equivariant homeomorphism (allowing an automorphism of S^1 if H reverses the orientation) by [13] and [17]. Consequently, the restriction of the equivariant homeomorphism to (S^1, X) is the desired map.

We used the generalized Nielsen theorem to obtain 9.3 (in the closed case). It is not hard to see, using [6], that 9.3 implies the generalized Nielsen theorem. This suggests that we may use the classification of S^1-actions to yield a classification of planar groups with non-compact quotients. This is actually the case. In fact, as we shall see, one only needs to use the more elementary equivariant classification.

Let N be finitely generated, non-cyclic and be isomorphic to an orientation preserving group of properly discontinuous groups of homeomorphisms of the plane with non-compact quotient. Then N is isomorphic to $Z * \ldots * Z * Z_{\alpha_1} * \ldots * Z_{\alpha_n}$, and satisfies 9.6. Furthermore, R^2/N has no boundary. The number of free factors is equal to $2g+h-1$. There are clearly $g+1$ distinct orbit spaces, all of which are possible. Let us fix one orbit space of genus g. We wish to determine all possible equivalence classes of actions of N on R^2 whose orbit space has genus g when it is completed by the addition of exactly h points.

All possible (S^1, X) which has R^2/N as orbit space and $\{\alpha_1, \ldots, \alpha_n\}$ as the orders of all of its singular orbits (allowing repetitions of course) are given by

$$(S^1 X) \sim \{\epsilon; (g, 0, h); (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\} .$$

Now each of these may be obtained by taking a fixed (R^2, N) with the desired orbit space of genus g and choosing a pre-extension in the sense of [6 ; § 9 and 10]. That is, we choose a section $H^0(R^2/N; \underset{\sim}{h}^2) \cong H^2(N; Z) \cong Z_{\alpha_1} \oplus \ldots \oplus Z_{\alpha_n}$ of the sheaf $\underset{\sim}{h}^2$ so that the projection to each Z_{α_i} yields a generator ν_j (reduced mod α_j). Now $\beta_j \nu_j \equiv 1 \bmod \alpha_j$ and therefore each action of N on R^2 yields all of the manifolds (S^1, X) by just choosing different Bieberbach classes in $H^2(N; Z)$. Suppose (S^1, X) and (S^1, X') corresponding to a and a' are equivariantly homeomorphic, then by [6 ; 8.6] there exists an automorphism $\Phi: N \longrightarrow N$ and a homeomorphism $\Psi: R^2 \longrightarrow R^2$ so that $\Phi^*(a') = a$ and $\Psi(w\alpha) = \Psi(w)\Phi(\alpha)$.

Now let (R^2, N') denote an action perhaps different from (R^2, N) but with the same orbit space and $N \cong N'$. Then (S^1, X) is represented by some class $b \in H^2(N'; Z)$. Thus, as (S^1, X) is also represented by $a \in H^2(N; Z)$ this means, via [6 ; 8.6], that there exists an automorphism of $\Phi': N' \longrightarrow N'$, so that the actions $(R^2, \Phi'(N'))$ and (R^2, N) are equivariantly homeomorphic. Thus we have a non-compact form of the generalized Nielsen theorem.

9.8 Corollary: Let N be an orientation preserving, effective, properly discontinuous, finitely generated, non-cyclic planar group whose quotient space is not compact. Then N is isomorphic to $Z * \ldots * Z * Z_{\alpha_1} * \ldots * Z_{\alpha_n}$ where there are m copies of Z. The distinct actions of N on R^2 up to orientation preserving equivariant homeomorphism are in one to one correspondence with the distinct unordered sets

$$\left\{\epsilon; (g, h, 0); (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\right\}$$

where $2g+h-1 = m$.

10. The 3-dimensional examples

Let (S^1, X) be an effective action on a connected 3-manifold with finitely generated fundamental group. Assume that it fibers, equivariantly, over the circle $(S^1, S^1/Z_m)$ with structure group Z_m and 2-manifold Y as fiber. We can assume without loss of generality that Y is connected. When X is orientable we may also assume that no subgroup of Z_m acts freely. For, suppose $Z_k \subseteq Z_m$ acts freely on Y, then $(S^1, S^1 \times_{Z_k} Y)$ is a principal circle action whose characteristic class is an element of $H^2(\pi_1(Y/Z_k); Z)$. Since $(S^1, S^1 \times_{Z_k} Y)$ fibers over S^1/Z_k, this characteristic class must be of finite order and hence must be 0. Thus $(S^1, S^1 \times_{Z_k} Y)$ is S^1-equivariantly homeomorphic to $(S^1, S^1 \times Y/Z_k)$. (Notice the different fiberings.)

An invariant tubular neighborhood of $x \in X$ is given by $(S^1, S^1 \times_{Z_{\alpha_j}} D_x)$, where Z_{α_j} is the stability group at x, and D_x the 2-disk slice at x. When we lift the action to $(S^1, S^1 \times Y)$ then the slice lifts to the slice D at (t, y) where $((t, y)) = x$. The group Z_m translates the disk D in Y into $m/\alpha_j = r$ distinct disks. Of course the isomorphism $\eta_x: Z_{((t,y))} \longrightarrow (Z_m)_y$ of [6; §4] induces the same representation of Z_{α_j} on D.

Thus let us begin with T a generator of the action of Z_m on Y. We choose representatives $y_1, \ldots, y_n$ in Y of the distinct orbits with stability groups $Z_{\alpha_1}, \ldots, Z_{\alpha_n}$. The slice representations, when there are no reflections, are equivalent to

$$\rho e^{i\theta} \longrightarrow \rho e^{i\theta} \exp\left(\frac{2\pi i \nu_j}{\alpha_j}\right).$$

The group generated by T^{r_j}, isomorphic to Z_{α_j}, yields this representation. We shall call the manifold $(S^1, X) = (S^1, S^1 \times_{Z_m} Y)$ by $(S^1, X(1))$.

When Y is closed and oriented, we represent $(S^1, X(1))$ by its complete set of oriented orbit invariants

$$\{\epsilon; g; b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\} \quad .$$

Now let us construct $X(q)$. This means we take the automorphism $Z_m \longrightarrow Z_m$ which sends T to T^q, $(m, q) = 1$. The slice representation at y_j will be altered to the representation given by $T^{r_j q}$. That is

$$\rho e^{i\theta} \longrightarrow \rho e^{i\theta} \exp\left(\frac{2\pi i \nu_j}{\alpha_j}\right)$$

will be altered to

$$\rho e^{i\theta} \longrightarrow \rho e^{i\theta} \exp\left(\frac{2\pi i q \nu_j}{\alpha_j}\right) \quad .$$

Since $\beta_j \nu_j \equiv 1 \bmod \alpha_j$, and $\beta_j' q \nu_j \equiv 1 \bmod \alpha_j$, $\beta_j' \equiv \beta_j q^{-1} \bmod \alpha_j$. Thus our new orbit invariants representing $(S^1, X(q)) = (S^1, S^1 \times_{Z_n^{(2)}} Y)$ will be

(10.1) $$\{\epsilon; g, b'; (\alpha_1, \beta_1 q^{-1}), \ldots, (\alpha_n, \beta_n q^{-1})\}$$

Of course the decision as to whether $X(1)$ is homeomorphic to $X(q)$ is given by 9.3, provided that 9.2 is satisfied.

To what extent we may find examples is dependent upon our being able to recognize when the action represented by $\{\epsilon; g; b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ represents an action which fibers equivariantly over the circle. In [8; 12.13] we found that

10.2. Lemma: If (S^1, M^3) is closed and oriented, then (S^1, M^3) fibers equivariantly over the circle if and only if

$$b + \frac{\beta_1}{\alpha_1} + \ldots + \frac{\beta_n}{\alpha_n} = 0 \, .$$

In particular, the sum of the irreducible fractions

$$\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \ldots + \frac{\beta_n}{\alpha_n}$$

must be integral.

Notice for any choice of $(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)$ where $\beta_1/\alpha_1+\ldots+\beta_n/\alpha_n$ is integral, the integer b is determined if (S^1, M^3) is to fiber with finite structure group. Also the genus of the orbit space does not matter. We have obtained the following

10.3. Theorem: Let $\{(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\}$ be a set of Seifert invariants so that

$$\frac{\beta_1}{\alpha_1}+\ldots+\frac{\beta_n}{\alpha_n}$$

is integral. Let q be any integer which is relatively prime to the least common multiple m of $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. Put $-b=\beta_1/\alpha_1+\ldots+\beta_n/\alpha_n$ and choose any $g\geq 0$. Form $(S^1,M^3)=(S^1,X(1))$ whose equivariant homeomorphism type is given by $\{\epsilon; g; b; (\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\}$. Then $X(q)$ is homeomorphic to $X(1)$ if and only if the unordered sets

$$\{(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\} \quad \text{or} \quad \{(\alpha_1,\alpha_1-\beta_1),\ldots,(\alpha_n,\alpha_n-\beta_n)\}$$

is the same as

$$\{(\alpha_1,\beta_1q^{-1}),\ldots,(\alpha_n,\beta_nq^{-1})\}$$

where β_jq^{-1} is reduced modulo α_j.

10.4. Here are a few illustrative examples.

(i). $$\left(S^1,X(1)\right) = \{\epsilon; g; -2; (2,1),(4,3),(8,1),(8,1),(8,1),(8,3)\}$$

Let us take Z_8 and form the automorphism $T\longrightarrow T^3$. Under this automorphism $\beta_i\longrightarrow\beta_i3^{-1}$. Now, $3^{-1}\equiv 3 \bmod 8$, $3^{-1}\equiv 1 \bmod 2$, $3^{-1}\equiv 3 \bmod 4$, and so,

$$\left(S^1,X(3)\right) = \{\epsilon; g; -2; (2,1),(4,1),(8,3),(8,3),(8,3),(8,1)\} \quad .$$

Note that the oppositely oriented

$$-\left(S^1,X(1)\right) = \{\epsilon; g; -4; (2,1),(4,1),(8,7),(8,7),(8,7),(8,5)\} \quad .$$

Thus $\pi_1(X(1)) \not\cong \pi_1 X(q)$. In this example it is not very easy to calculate what oriented surface Y we are dealing with.

(ii). In our next class of examples it is clear what surface Y we are dealing with. In fact, the examples are the same as those illustrated in §8. For each prime $p\geq 3$, there is a Riemann surface of genus $((p-1)(p-2))/2$ on which Z_p acts with exactly p fixed points, and with quotient space the Riemann sphere. In terms of its orbit invariants it is described by

$$M(j^{-1}) = \{\epsilon; ((p-1)(p-2))/2; -j; (p,j),\ldots,(p,j)\}$$

with p copies of (p, j). Here it is obvious that $\pi_1(M(1)) \not\approx \pi_1(M(j^{-1}))$, for $1 < j < p-1$. For fixed g, there are $(p-1)/2$ mutually topologically distinct spaces in this case.

(iii). Another manifold is

$$M(1) = \left\{\epsilon; g; -\frac{p-1}{2}; (p,1), (p,2), (p,3), \ldots, (p, p-1)\right\}.$$

Notice that $M(1) = M(q)$, for all q.

Obviously 10.3 is algorithmic and topologically distinct $X(1)$ and $X(q)$ have distinct fundamental groups. For non-closed manifolds, as we have seen, the fundamental groups may be isomorphic while the manifolds (S^1, X) and (S^1, X') are topologically distinct.

10.5. Let us treat the non-closed oriented case, (S^1, X). Each such action fibers equivariantly over the circle since every action is homologically injective. Let us assume that 9.6 is satisfied as well as the boundary of X is compact (but perhaps empty). We may write $(S^1, X) = (S^1, X(1)) = (S^1, S^1 \times_{Z_m} Y)$. Y is an oriented 2-manifold on which Z_m operates so that no subgroup operates freely. The orbit space $Y/Z_m = X/S^1$ is a 2-manifold with $s \geq 0$ boundary components, h holes and genus g. There are n-marked points $\{y_1^*, \ldots, y_n^*\} = E$, corresponding to singular orbits with stability groups $(\alpha_1, \ldots, \alpha_n)$. By choosing T as generator of Z_n we may write

$$\left(S^1, X(1)\right) = \left\{\epsilon, (g, h, s); (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\right\} \quad .$$

All such representations in contradistinction to the closed case are possible. Just as before

$$\left(S^1, X(q)\right) = \left\{\epsilon; (g, h, s); (\alpha_1, \beta_1 q^{-1}), \ldots, (\alpha_n, \beta_n q^{-1})\right\}.$$

10.6. Theorem: X(1) is homeomorphic to X(q) if and only if the unordered set

$$\left\{(\alpha_1, \beta_1 q^{-1}), \ldots, (\alpha_n, \beta_n q^{-1})\right\}$$

equals the unordered sets

$$\left\{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\right\} \quad \text{or} \quad \left\{(\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_n, \alpha_n - \beta_n)\right\} \quad .$$

Of course, by $(\alpha_j, \beta_j q^{-1})$ we mean (α_j, β_j'), where $\beta_j' \equiv \beta_j q^{-1} \bmod \alpha_j$ and $0 < \beta_j < \alpha_j$. One other distinguishing characteristic of $X(q)$ from $X(1)$ if they are compact but not homeomorphic is that for any map $f: X(1) \longrightarrow X(q)$ inducing an isomorphism $f_*: \pi_1(X(1), x_1) \longrightarrow \pi_1(X(q), x_2)$, the image of

$$i_*:\pi_1\big(\partial(X(1))\big) \longrightarrow \pi_1\big(X(1)\big)$$

by f_* is not conjugate to the image of

$$i_*:\pi_1\big(\partial(X(q))\big) \longrightarrow \pi_1\big(X(q)\big) \quad .$$

That is, their peripheral structures are not the same.

An interesting identification can be made for

$$\big(S^1, M(1)\big) = \big\{\epsilon; (g,h,0); (\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\big\} \quad .$$

We compactify equivariantly to a closed oriented manifold instead of a compact manifold by collapsing each orbit of the added toral boundaries of 9.6 to a point. This adds, instead of a torus, just a circle. The action extends so all point of the added circles appear as fixed points. The compactified manifold may easily be identified from [14] to be

$$S^3 \# (S^2 \times S^1)_1 \# \ldots \# (S^2 \times S^1)_{2g+h-1} \# L(\alpha_1,\nu_1) \# \ldots \# L(\alpha_n,\nu_n) \ .$$

$\#$ denotes (equivariant) connected sum of these $S^2 \times S^1$'s with lens spaces. The manifold $\big(S^1, M(1)\big)$ is obtained by just deleting the fixed point set of h circles. Note that the equivariant classficiation of $\big(S^1, M(1)\big)$ and of this type of compactification coincide.

As an illustration, let a and b be relatively prime. The complement, in the 3-sphere, of a torus knot $K(a,b)$ is

$$M(a,b) = \big(\epsilon; (0,1,0); (a,\tilde{b}), (b,\tilde{a})\big\} \quad , \tag{10.7}$$

where $\tilde{b}^{-1} \equiv b \bmod a$ and $\tilde{a}^{-1} \equiv a \bmod b$. This arises from removing the circle of fixed points in the space $L(a,\tilde{b}^{-1}) \# L(b,\tilde{a}^{-1})$. The cyclic group Z_{ab} operates on the surface Y of genus $\big((a-1)(b-1)\big)/2$ with one hole (the spanning surface of the knot) and $M(a,b) = S^1 \times_{Z_{ab}} Y$. If we regard $X(1) = M(a,b)$, then

$$X(q) = \big\{\epsilon; (0,1,0); (a,\tilde{b}'), (b,\tilde{a}')\big\}$$

where $\tilde{b}' \equiv \tilde{b}q^{-1} \bmod a$, $\tilde{a}' \equiv \tilde{a}\, q^{-1} \bmod b$. For example, let $a = 5$, $b = 7$, then we have orbit invariants for $X(1)$, $\{(5,3),(7,3)\}$. If we take $X(2)$ we get orbit invariants $\{(5,4),(7,5)\}$. Thus $X(1)$ and $X(2)$ are not equivariantly homeomorphic, hence not homeomorphic. (Furthermore, $X(2)$ is not the complement of any knot.) Their fundamental groups are isomorphic of course. Note that $L(5,2) \# L(7,5)$ (whose orbit invariants are $\{(5,3),(7,3)\}$) is not homeomorphic to $L(5,4) \# L(7,5)$ (whose orbit invariants are $\{(5,4),(7,5)\}$.

10.8. In [8 ; 12] we pointed out a connection between $S^1 \times X(q)$ and projective non-singular algebraic surfaces, when $X(q)$ is closed and oriented. Now the reader familiar with [8] will easily see that the entire 3-dimensional construction can be done in the complex analytic category if we replace S^1 by C^*, the multiplicative group of complex numbers.

Theorems 10.6, 10.3 allow us to make another interesting observation. Let Z_m act on an oriented surface Y. In

$$\left(S^1, X(1)\right) = (S^1, S^1 \times_{Z_m} Y)$$

there exists $Z_r \subseteq S^1$, with $rq \equiv 1 \bmod m$ so that $\left({}^1S^1, X(q)\right)$ is equivariantly homeomorphic to $(S^1/Z_r, X(1)/Z_r)$. In particular, <u>if</u> $X(q)$ is homeomorphic to $X(1)$, then $X(1)$ <u>is a regular r-fold cyclic (unbranched) covering of itself in addition to the rq-fold covering of</u> §1 <u>and</u> §4.

We also mention that almost everything also carries over to the non-orientable case including even our non-compact version of the generalized Nielsen theorems. The changes needed are minor except when the properly discontinuous group contains reflections.

11. Products of Charlap actions

In Sections 8 and 10 we have given a large class of examples of 3-dimensional closed manifolds for which $S^1 \times M^3(1) = S^1 \times M^3(q)$, but $\pi_1\left(M^3(1)\right) \neq \pi_1\left(M^3(q)\right)$. Earlier in 6.6 and 7.1 we exhibited actions on fairly high dimensional closed manifolds with the above property. In certain cases it is possible to form a product of two Charlap actions to obtain a third. These constructions are useful in presenting other examples. We shall discuss two of these procedures here. In the first we restrict ourselves to prime periods. These lead to fairly sophisticated Charlap actions for which the trace invariant can be used to detect non-isomorphic fundamental groups arising from 4-dimensional Charlap actions, (11.3).

In the second type of construction (11.4), we allow periodic maps of composite period. We detect non-isomorphic fundamental groups by 6.4. These examples arise in all dimensions greater than 2.

11.1. We suppose given two diffeomorphisms of prime period, p, (T_1, M_1, x_1) and (T_2, M_2, x_2) where

(a) x_i is a fixed point

(b) M_i is a closed aspherical manifold

(c) $H^1(M_i/T_i; Q) = 0$.

Now in addition we shall also suppose that there is a fixed point free diffeomorphism of

period p, $\tau : M_1 \longrightarrow M_1$ which commutes with T_1. We form the space $M = (M_1 \times M_2)/(\tau \times T_2)$ by identifying (x, y) with $\left(\tau^j(x), T_2^j(y)\right)$ for $0 \le j < p$. Denoting by $((x, y)) \in M$ a point in M we introduce a map of period p, (σ, M) by

$$\sigma((x, y)) = \left(\left(T_1(x), y\right)\right)$$

which is well defined since $T_1\tau = \tau T_1$. Of course M is a closed aspherical manifold for which there is a fibration $\nu : M \longrightarrow M_1/\tau$ with fiber M_2 and structure group Z_p acting by T_2. Furthermore, if $(T', M_1/\tau)$ is induced by T_1, then $\nu : (\sigma, M) \longrightarrow (T', M_1/\tau)$ is also equivariant. Surely (σ, M) has at least one fixed point, $((x_1, x_2))$. We must prove that $H^1(M/\sigma; Q) = 0$.

First let us remark $H^1\left((M_1/\tau)/T'; Q\right) = 0$. Observe $H^1(M_1/\tau; Q)$ is naturally isomorphic to the kernel of $I - \tau^* : H^1(M_1; Q) \longrightarrow H^1(M_1; Q)$. Call this kernel V. Since $T_1\tau = \tau T_1$, V is T_1^*-invariant, and $I - T'^* : H^1(M_1/\tau; Q) \longrightarrow H^1(M_1/\tau; Q)$ may be identified with $I - T_1^* : V \longrightarrow V$. Since $I - T_1^* : H^1(M_1; Q) \simeq H^1(M_1; Q)$ it follows $H^1\left((M_1/\tau)/T'; Q\right) = 0$ also. It will thus be sufficient to show the equivariant $\nu : (\sigma, M) \longrightarrow (T', M_1/\tau)$ induces an isomorphism $\nu^* : H^1(M_1/\tau; Q) \simeq H^1(M; Q)$. We may regard $H^1(M; Q)$ as the kernel of

$$I - (\tau \times T_2)^* : H^1(M_1 \times M_2; Q) \longrightarrow H^1(M_1 \times M_2; Q) .$$

If, by the Künneth formula we write

$$H^1(M_1 \times M_2; Q) = H^1(M_1; Q) \otimes H^0(M_2; Q) \oplus H^0(M_1; Q) \otimes H^1(M_2; Q)$$

$$= H^1(M_1; Q) \oplus H^1(M_2; Q) \quad ,$$

then on the first summand $(\tau \times T_2)^*$ is τ^* so $V \subset H^1(M_1; Q)$ lies in $\ker\left(I - (\tau \times T_2)^*\right)$ while on the second summand $(\tau \times T_2)^*$ becomes T_2^* and $I - T_2^*$ is an automorphism. Thus with the aid of the section $\chi : M_1/\tau \longrightarrow M$ we see $\nu^* : H^1(M_1/\tau; Q) \simeq H^1(M; Q)$. Therefore $H^1(M/\sigma; Q) = 0$ and (σ, M) is a Charlap action.

Since $\nu : M \longrightarrow M/\tau$ admits a cross-section, we may express $\pi_1(M)$ as a semi-direct product $\pi_1(M_2) \circ \pi_1(M_1/\tau)$. The homomorphism $\pi_1(M/\tau) \longrightarrow \mathrm{Aut}\left(\pi_1(M_2)\right)$ is the composition of $Z_p \longrightarrow \mathrm{Aut}\left(\pi_1(M_2, x_2)\right)$ given by (T_2, M_2, x_2) and $1 \longrightarrow \pi_1(M_1) \longrightarrow \pi_1(M_1/\tau) \longrightarrow Z_p \longrightarrow 0$.

We may also analyze the fixed point set of (σ, M) as follows. First for each j, $0 < j < p$ denote by $C_j \subset M_1$ the set of all $x \in M_1$ with $\tau(x) = T^j(x)$. Thus C_j is a set of coincidences, which may be empty. If we choose i, $0 < i < p$ with $ij \equiv -1 \bmod p$ then C_j is also the fixed point set of $T\tau^i$, a diffeomorphism of period p on M_1. If $j \neq j'$ then

$C_j \cap C_{j'} = \phi$, for if this were false then

$$T\tau^{i}(x) = x = T\tau^{i'}(x)$$

implying $\tau^{i-i'}(x) = x$, but τ is fixed point free. Let $F_1 \subset M_1$ and $F_2 \subset M_2$ be the fixed point set of (T_1, M_1) and (T_2, M_2). The fixed point set of (σ, M) is a disjoint union

$$(F_1 \times M_2)/(\tau \times T_2) \sqcup (C_1/\tau \times F_2) \sqcup \ldots \sqcup (C_{p-1}/\tau \times F_2) .$$

We note $(F_1 \times M_2)/(\tau \times T_2)$ fibers over F_1/τ with fiber M_2 and structure group Z_p.

Naturally the process can be iterated by combining (T_1, M_1) with (σ, M) via τ again. Perhaps the simplest example is found by taking $(T_1, M_1) = (T_2, M_2)$ to be the involution on S^1 given by $z \to \bar{z}$ and $\tau(z) = -z$. Then M is the Klein bottle and by iteration we obtain the standard generalization of K^2.

A rather puzzling example can be given as follows. Let (T, M_1) be the automorphism of the $(p-1)$-torus given by $A(z_1, \ldots, z_{p-1}) = (z_2, z_3, \ldots, z_{p-1}, z_1^{-1}, \ldots, z_{p-1}^{-1})$, which has period p. If $\lambda = \exp(2\pi i/p)$ then $A(\lambda^j, \ldots, \lambda^j) = (\lambda^j, \ldots, \lambda^j, (\lambda^{-j})^{p-1}) = (\lambda^j, \ldots, \lambda^j)$. The points $(\lambda^j, \ldots, \lambda^j)$, $0 < j < p-1$ make up the fixed point set $F_1 \subset M_1$. Now suppose $t * \lambda = (z_1 \lambda, \ldots, z_{p-1}\lambda)$, then $\tau(t) = t * \lambda$ commutes with A, since A is an automorphism leaving $(\lambda, \ldots, \lambda)$ fixed. Obviously τ is fixed point free since it is multiplication by λ in each co-ordinate. For each i, $0 < i < p$, we consider the fixed point set, C_j, of

$$A\tau^{i}(z_1, \ldots, z_{p-1}) = (z_2\lambda^{i}, \ldots, z_{p-1}\lambda^{i}, z_1^{-1} \ldots z_{p-1}^{-1}\lambda^{i})$$

which is exactly the set of p elements given by $(\lambda^{p-i+k}, \lambda^{2(p-i)+k}, \ldots, \lambda^{(p-1)(p-i)+k})$ $0 \le k < p$. Note τ freely permutes the points in $F_1; C_1, \ldots, C_{p-1}$. Now for (T_2, M_2) we take an automorphism on the $(p-1)$-torus corresponding to a non-trivial ideal equivalence class. When (σ, M) is formed we see $(C_j/\tau) \times F_2$ is a set of p-points for each j, $0 < j < p$ while $(F_1 \times M_2)/(\tau \times T_2) = M_2$, the fiber of ν over the point $\nu(F_1) \in M_1/\tau$. Thus there are $p(p-1)$ isolated fixed points. For this example, however, we have no idea what may be said about the associated $X(q)$ which may be formed. We have no idea, in fact, about even beginning.

11.2. We shall for each odd prime p present a 4-dimensional Charlap action (Z_p, M^4, x_0). We shall compute the invariant $Tr(Z_p, M^4)$ by analyzing the fixed point set.

We recall the Charlap action (Z_p, S) on the algebraic curve $S \subset CP(Z)$ of genus $(p-1)(p-2)/2$ given in homogeneous co-ordinates by

$$S = \left\{ [z_1, z_2, z_3] \,\middle|\, z_1^p + z_2^p + z_3^p = 0 \right\}$$

$$T[z_1, z_2, z_3] = [\lambda z_1, z_2, z_3] .$$

The quotient space S/T was identified with $CP(1)$. Now let us introduce a second map of period p on S, commuting with T, by

$$\tau[z_1, z_2, z_3] = [\lambda^{-1} z_1, \lambda z_2, \lambda z_3].$$

We claim τ is fixed point free, for if for some $\mathfrak{z} \in C^*$

$$(\mathfrak{z} z_1, \mathfrak{z} z_2, \mathfrak{z} z_3) = (\lambda^{-1} z_1, \lambda z_2, z_3)$$

then either $z_3 \neq 0$, in which case $\mathfrak{z} = 1$ and $z_1 = z_2 = 0$ and that is impossible, or else $z_3 = 0$ in which case neither z_1 nor z_2 vanish and $\lambda = \mathfrak{z} = \lambda^{-1}$, another contradiction. So we may form $Y = S/\tau$, which is a closed oriented surface of genus $(p-1)/2$ on which T, since it commutes with τ, induces a map of period p, (T_1, Y), for which $Y/T_1 = CP(1)$ again, of course. We wish to prove (T_1, Y) always has exactly 3 fixed points. Recall first the fixed point set $F \subset S$ of (T, S) which consists of p points: $\left\{ [0, \lambda^j, -1] \right\}_{j=0}^{p-1}$. Next if $0 \quad j \quad p$ the maps τ and T^j may have coincidences. Suppose $\tau[z_1, z_2, z_3] = T^j[z_1, z_2, z_3]$, then $\lambda^{-1} z_1 = \mathfrak{z} \lambda^j z_1$, $\lambda z_2 = \mathfrak{z} z_2$, $z_3 = \mathfrak{z} z_3$, for some $\mathfrak{z} \in C^*$. If $z_3 \neq 0$, then $\mathfrak{z} = 1$, $z_2 = 0$ and $\lambda^{-1} z_1 = \lambda^j z_1$ implies $j = p-1$. Thus τ and T^{p-1} have as their set of coincidences $C_1 = \left\{ [\lambda^j, 0, -1] \right\}_{j=0}^{p-1}$. If on the other hand $z_3 = 0$, then neither z_1 nor z_2 is zero and so $\mathfrak{z} = \lambda$ and $\lambda^{-1} = \lambda^{j+1}$ which implies $j = p-2$. Thus τ and T^{p-2} have as their set of coincidences $C_2 = \left\{ [\lambda^j, -1, 0] \right\}_{j=0}^{p-1}$. Clearly τ freely and cyclicly permutes the points of each of the three sets F, C_1, C_2 so that the image of these three sets in Y is just three points y_0, y_1 and y_2. Obviously these are just the fixed points of (T_1, Y). We note also that τ and T^j have a non-empty set of coincidences if and only if $j = p-1$ or $p-2$ according to our argument.

Now C_1 may clearly be regarded as the fixed point set of $T\tau$. Also, if we note that $1-(p+1)/2 = (p+1)/2 - p$, then C_2 is the fixed point set of $T\tau^{p+1/2}$. Also

$$T\tau^{p+1/2}[z_1, z_2, z_3] = [\lambda^{1-(p+1)/2} z_1, \lambda^{(p+1)/2} z_2, z_3]$$

$$= [\lambda^{p+1/2} z_1, \lambda^{p+1/2} z_2, z_3] = [z_1, z_2, \lambda^{p-1/2} z_3].$$

Now the quotient map is an equivariant local diffeomorphism of degree $+p$ in each of the three cases

$$(T, S) \longrightarrow (T_1, Y)$$

$$(T\tau, S) \longrightarrow (T_1, Y)$$

$$(T\tau^{p+1}, S) \longrightarrow (T_1, Y) \quad .$$

The problem is to determine at each of the points y_0, y_1 and y_2 the local 1-dimensional representation of Z_p in the complex tangent line at the point. But then we can do this by selecting points in S fixed under T, $T\tau$ and $T\tau^{p+1/2}$ and finding the corresponding local representation in the tangent line to S at these points. For T we have already agreed that the local representation at a fixed point is multiplication by λ. Now $T\tau\left[z_1, z_2, z_3\right] = \left[z_1, \lambda z_2, z_3\right]$, however $\left[z_1, z_2, z_3\right] \longrightarrow \left[z_2, z_1, z_3\right]$ is an orientation preserving equivariant diffeomorphism of $(T\tau, S)$ with (T, S), hence the local representation at y_1 is also multiplication by λ. Finally, $T\tau^{p+1/2}\left[z_1, z_2, z_3\right] = \left[z_1, z_2, \lambda^{p-1/2} z_3\right]$ and $\left[z_1, z_2, z_3\right] \longrightarrow \left[z_3, z_2, z_1\right]$ is an equivariant equivalence between $T\tau^{p+1/2}$ and $T^{p-1/2}$, hence the local representation at y_2 is multiplication by $\lambda^{p-1/2}$.

We come now to M^4. On $S \times S$ we identify (s_1, s_2) with $\left(\tau^j(s_1), T^j(s_2)\right)$ for all $0 \leq j < p$. The identification is M^4 which is a closed oriented aspherical manifold fibered over Y with fiber S and structure group Z_p. Since (T, S) has a fixed point, $M^4 \longrightarrow Y$ admits a cross-section and $\pi_1(M^4)$ is a semi-direct product $\pi_1(S) \circ \pi_1(Y)$.

Denoting by $((s_1, s_2))$ a point in M^4 we now introduce the map of period p by $\sigma((s_1, s_2)) = ((Ts_1, s_2))$. Let us now work out the fixed point set of σ. First of all, the set $F \times S \subset S \times S$ under the identification map $S \times S \longrightarrow M^4$ is obviously carried into the fixed point set of (σ, M^4). In fact, its image is just the fiber in $M^4 \longrightarrow Y$ over the point y_0. Thus we have established the fixed point set contains a copy of S and, most important, it has a product (trivial) normal bundle in M^4 because it is a fiber.

The other fixed points are now exhibited as follows. If $((s_1, s_2))$ is fixed under σ, but $s_1 \notin F$ then for some j, $0 < j < p$

$$\tau s_1 = T^j s_1 , \quad Ts_2 = s_2 \quad .$$

Thus $s_2 \in F$ and $s_1 \in C_1 \cup C_2$. Thus the rest of the fixed point set is $(C_1/\tau) \times F \cup (C_2/\tau) \times F$ and so it consists of $2p$ isolated distinct points. At each of these we receive a local representation of Z_p in C^2. We proceed as in the case of (T_1, Y). Thus we have $S \times S \longrightarrow M^4$ an equivariant local diffeomorphism

$$(T\tau \times T, \ S \times S) \longrightarrow (\sigma, M^4)$$

$$(T\tau^{p+1/2} \times T^{p+1/2}, \ S \times S) \longrightarrow (\sigma, M^4) \quad .$$

It will be noted that the assertion follows since $\left(\!\left(T\tau(s_1), T(s_2)\right)\!\right) = \left(\!\left(\tau T(s_1), T(s_2)\right)\!\right)$ $= \left(\!\left(T(s_1), s_2\right)\!\right) = \sigma((s_1, s_2))$ and similarly $\left(\!\left(T\tau^{p+1/2}(s_1), T^{p+1/2}(s_2)\right)\!\right) = \left(\!\left(T(s_1), s_2\right)\!\right)$ $= \sigma((s_1, s_2))$. But the fixed point sets of $T\tau \times T$ and $T\tau^{p+1/2} \times T^{p+1/2}$ are respectively $C_1 \times F$ and $C_2 \times F$. Hence at each point of $C_1/\tau \times F$ the local representation of Z_p is (λ, λ) while at each point in $C_2/\tau \times F$ it is

$$(\lambda^{p-1/2}, \lambda^{p+1/2}) = (\lambda^{p-1/2}, \bar{\lambda}^{p-1/2}) .$$

At this point we must introduce a little bordism to show that the positive dimensional component of the fixed point set of (Z_p, M^4) can be ignored in computing $\mathrm{Tr}(Z_p, M^4)$. We recall that $\mathrm{Tr}(Z_p, M^4)$ depends only on fixed point data; that is, on the image of the oriented bordism class $\left[Z_1, M^2\right] \in \mathcal{O}_4^{SO}(Z_p)$, the oriented bordism group of unrestricted Z_p-actions, under the homomorphism $\mathcal{O}_4^{SO}(Z_p) \longrightarrow \mathcal{M}_4$. This sends $\left[Z_p, M^4\right]$ into the bordism classes of boundary free Z_p-actions, $\mathcal{M}_4$, by assigning to $\left[Z_p, M^4\right]$ the boundary free bordism class of the Z_p-action on a closed invariant normal tube around the fixed point set. Now the fiber $S \subset M^4$ which is fixed has a trivial normal bundle, hence its image in $\mathcal{M}_4$ is $\left[S\right] \cdot \left[Z_p, D^2\right]$ where D^2 is the closed 2-cell. But $\Omega_2^{SO} = 0$, thus $\left[S\right] \cdot \left[Z_p, D^2\right] = 0$. Therefore $\mathrm{Tr}\left[Z_p, M^4\right]$ is computed only by applying $Le: \mathcal{M}_4 \longrightarrow Q(\lambda)$ at the isolated fixed points. Since $p - (p+1)/2 = (p-1)/2$, we may thus choose the orientation of M^4 so that we have

11.3. <u>Theorem</u>:

$$\mathrm{Tr}(Z_p, M^4) = p\left[\left(\frac{1+\lambda}{1-\lambda}\right)^2 - \left(\frac{1+\lambda^{(p-1)/2}}{1-\lambda^{(p-1)/2}}\right)^2\right]$$

Note that $\mathrm{Tr}(Z_p, M^4) \neq 0$, if $p > 3$.

11.4. We shall now exhibit some Charlap actions of composite period. Let (Z_m, Y_1) and (Z_n, Y_2) be Charlap actions such that $(m, n) = 1$. We form the closed aspherical manifold $Y = Y_1 \times Y_2$. On Y we define an action of Z_{mn} by defining the map $T: Y \longrightarrow Y$ of period mn by

$$T(y_1, y_2) = (T_1 y_1, T_2 y_2) ,$$

where $T_1: Y_1 \longrightarrow Y_1$, $T_2: Y_2 \longrightarrow Y_2$ are the periodic maps generating the Charlap actions (Z_m, Y_1) and (Z_n, Y_2).

We factor the orbit map

$$(Z_{mn}, Y) \longrightarrow Y/Z_{mn}$$

into

$$(Z_m \times Z_n, Y) \xrightarrow{/Z_m} (Z_n, Y/Z_m) \xrightarrow{/Z_n} Y/Z_{mn} .$$

Notice that (T^n), which generates Z_m, acts trivially on the second factor Y_2. Thus $Y/Z_m = (Y_1/Z_m) \times Y_2$. The induced action of Z_n on Y/Z_m acts trivially on (Y_1/Z_m). Consequently, the orbit space

$$Y/Z_{mn} = (Y_1/Z_m) \times (Y_2/Z_n) .$$

Since both (Z_m, Y_1) and (Z_n, Y_2) have fixed points, (Z_{mn}, Y) has fixed points. Hence, (Z_{mn}, Y) is a Charlap action.

Let us now choose Y_1 to be the k-torus, T^k, $k \geq 1$. For Y_2 we choose any closed aspherical manifold, having a Charlap action (Z_n, Y_2), and such that $\pi_1(Y_2)$ has trivial center. (What is actually needed here is that (Z_n, Y_2) is an action so that $H_1(Y_2/Z_n; Q) = 0$, and that $\pi_1(Y_2)$ has trivial center. We do not need to assume that the space Y_2 is an aspherical manifold.) For example, any closed surface of genus $\big((p-1)(p-2)\big)/2$, p a prime greater than 3 will do. We let $Y = T^k \times Y_2$ and choose (Z_m, T^k) a Charlap action and choose q so that $(m, n) = 1$, $(mn, q) = 1$. (If $k = 1$, we let $m = 2$, if $k = 2$, then $m = 2, 3, 4, 6$ will do.) Now we further choose q so that $(T_2^q)_*$ is not conjugate to $(T_2^{\pm 1})_*$ in Out $\pi_1(Y_2)$. (Such examples exist of course for surfaces, products of surfaces, spaces realizing 6.6 and 7.1, the manifolds (Z_p, M^4, x) of 11.3, etc.)

Form the closed aspherical manifolds $X(1) = S^1 \times_{Z_{mn}} (T^k \times Y_2)$ and the corresponding $X(q)$.

11.5. Theorem: $S^1 \times X(1)$ is diffeomorphic to $S^1 \times X(q)$, but $\pi_1\big(X(1)\big)$ is not isomorphic to $\pi_1\big(X(q)\big)$.

Proof: Since $\big((T), T^k \times Y_2\big)$ is a Charlap action, we need only show, because of 6.4, that T_*^q is not conjugate to $T_*^{\pm 1}$ in Out $\pi_1(T^k \times Y_2)$.

In [9 ; 4.12] we found the split exact sequences:

$$0 \longrightarrow \mathrm{Hom}\big(\pi_1(Y_2), Z^k\big) \longrightarrow \mathrm{Aut}\big(\pi_1(T^k \times Y_2)\big) \rightleftarrows \mathrm{Aut}\big(\pi_1(Y_2)\big) \times \mathrm{Aut}(Z^k) \longrightarrow 1$$

$$0 \longrightarrow \mathrm{Hom}\big(\pi_1(Y_2), Z^k\big) \longrightarrow \mathrm{Out}\big(\pi_1(T^k \times Y_2)\big) \rightleftarrows \mathrm{Out}\big(\pi_1(Y_2)\big) \times \mathrm{Out}(Z^k) \longrightarrow 1 .$$

Therefore, if T_*^q is conjugate in Out $\pi_1(T^k \times Y_2)$ to $T_*^{\pm 1}$, then its image in $\mathrm{Out}(\pi_1(Y_2))$ must be conjugate to the image of $T_*^{\pm 1}$ in Out $\pi_1(Y_2)$. However, the image of T_*^q in Out $\pi_1(Y_2)$ is clearly $(T_1^q)_*$ in Out $\pi_1(Y_2)$ and that of $T_*^{\pm 1}$ is $(T_1^{\pm 1})_*$. Since we assumed the contrary, T_*^q must not be conjugate to $T_*^{\pm 1}$ in $\mathrm{Out}(\pi_1(T^k \times Y_2))$, and $\pi_1(X(1)) \not\cong \pi_1(X(q))$.

As an illustrative example, consider $S^1 = Y_1$, Y_2 to be a closed oriented surface of genus $((p-1)(p-2))/2$, p a prime greater than or equal to 7. Then $X(1) = S^1 \times_{Z_{2p}} (S^1 \times Y_2)$, a closed aspherical 4-manifold, and $(Z_{2p}, S^1 \times Y_2)$ is a 3-dimensional Charlap action. Of course, n doesn't have to be chosen prime. We may choose $n = \mathrm{lcm}(\alpha_1, \alpha_2, \ldots, \alpha_j)$ whenever $\beta_1/\alpha_1 + \ldots + \beta_k/\alpha_k$ is an integer. Then there is a surface Y_2 with (Z_n, Y_2) a Charlap action, and complex representations as we described in § 9. To use large m one must increase the dimension of the torus T^k. It should also be clear that $(Z_{mn}, T^k \times Y_2)$ can often be chosen as "algebraic", that is, $T^k \times Y_2$ admits the structure of a non-singular projective variety and that Z_{mn} acts as projective transformations.

While the process of 11.3 may be iterated to find more examples of Charlap actions, the space $T^k \times Y$ cannot be used again for 11.4 since $\pi_1(T^k \times Y)$ has non-trivial center.

12. Automorphisms of a semi-direct product

In §9 and §10 we indirectly exploited Theorem 5.5. Our treatment there was especially suited for the 3-dimensional examples. We shall develop in the next two sections higher dimensional analogues of these results, using bordism and the trace invariant. As in 9 and 10, we do not need to assume that our basic periodic maps are Charlap actions as in 8 and 11. Along with the generalized Nielsen theorem we were dealing in §9 and §10 with aspherical manifolds and groups whose quotients by their centers were centerless. These last two properties become relevant here.

Let us briefly outline the basic objective of this section. We restrict p to be an odd prime. We shall suppose we are given a cyclic group of orientation preserving diffeomorphisms (Z_p, Y, y_0) on a closed oriented aspherical 2k-manifold with at least one fixed point. We denote by $T: Y, y_0 \longrightarrow Y, y_0$ the diffeomorphisms of period p corresponding to the generator of Z_p. Of course T induces an automorphism T_* on $\pi_1(Y, y_0)$ with period p. Denoting this fundamental group by π and regarding Z_p as the cyclic group of p-th roots of unity, we introduce the semi-direct product $N = \pi \circ Z_p$ with multiplication

$$(\alpha, \lambda^i)(\beta, \lambda^j) = \left(\alpha T_*^i(\beta), \lambda^{i+j}\right) .$$

Then N acts on the (contractible) universal covering Y^* as a properly discontinuous group of orientation preserving diffeomorphisms with $Y^*/N = Y/Z_p$.

We have defined in [6] the subset $A \subset H^2(N; Z)$ corresponding to Baer classes of central extensions $0 \longrightarrow Z \longrightarrow L \longrightarrow N \longrightarrow 1$ in which the group L is torsionless. These classes are Bieberbach classes in the sense of [6]. We shall further restrict our attention to

$$A(p) = \left\{ a \,\middle|\, a \in A, \ \ pa = 0 \right\} .$$

We shall use the Atiyah-Bott fixed point theorem to define for each $a \in A(p)$ an invariant $tr(a) \in Z(\lambda)$. If $a, a' \in A(p)$ and if $\phi: N \longrightarrow N$ is an automorphism for which $\phi^*(a) = a'$ then we shall prove $tr(a') = \pm tr(a)$. This will yield a criterion used to distinguish elements in A(p) which are not equivalent under the natural action $\left(Out(N), H^2(N; Z)\right)$.

First there is a lemma about derived actions which must be presented. This will be a formula in $\mathcal{O}_{2k}^{SO}(Z_p)$, the group of unrestricted orientation preserving actions of Z_p on closed oriented 2k-manifolds (see [5] for details concerning this bordism group).

12.1. Lemma: If (Z_p, Y) is a group of orientation preserving diffeomorphisms on a closed oriented 2k-manifold, and if (Z_p, Y_f) is any derived action, then $[Z_p, Y] = [Z_p, Y_f] \in \mathcal{O}_{2k}^{SO}(Z_p)$.

Proof: We reconsider the definition of derived actions. Let $\nu: Y \longrightarrow Y/Z_p$ be the quotient and let $f: Y/Z_p \longrightarrow S^1$ be a map. Then we introduce $C \subset S^1 \times Y$, the set of all pairs (t, y) with $t^p f(\nu(y)) = 1$. On C we introduce

$$T_1(t,y) = (t, T_y)$$
$$T_2(t,y) = (t\lambda^{-1}, y) \ .$$

The quotient map $q:(T_1, C) \longrightarrow (T, Y)$, taken with respect to T_2 is a p-fold equivariant cyclic cover of Y. Thus C is a closed 2k-manifold oriented so that q has degree $+p$.

We wish to show $p[T,Y] = [T_1, C] \in \mathcal{O}^{SO}_{2k}(Z_p)$. We see immediately that $p[Y] = [C] \in \Omega^{SO}_{2k}$. We must apply the homomorphism

$$\mathcal{O}^{SO}_{2k}(Z_p) \longrightarrow \sum_{s=0}^{k}\Big(\sum_{n_1+\dots+n_{(p-1)/2}=s} \Omega^{SO}_{2(k-s)}\big(BU(n_1)\times\dots\times BU(n_{(p-1)/2})\big)\Big)$$

to compare the fixed point set data of (T, Y) and (T_1, C). The second sum is formed over all ordered $(p-1)/2$-tuples of non-negative integers with sum s. If $F \subset Y$ is the fixed point set of (T, Y) then F is a finite disjoint union of closed, connected regular submanifolds and $q^{-1}(F) \subset C$ is surely the fixed point set of (T_1, C). Now $F \subset Y$ has a normal bundle $\eta \longrightarrow F$, and, since p is odd this normal bundle may be given a complex structure and decomposed into a sum of complex bundles $\eta_1 \oplus \dots \oplus \eta_{(p-1)/2} \longrightarrow F$ corresponding to the eigenvalues $\lambda, \dots, \lambda^{(p-1)/2}$ of the bundle map on $\eta \longrightarrow F$ induced by T. But $q^{-1}(\eta) \longrightarrow q^{-1}(F)$ is the normal bundle in C, thus $q^{-1}(\eta_1) \oplus \dots \oplus q^{-1}(\eta_{(p-1)/2}) \longrightarrow q^{-1}(F)$ corresponds to the complex structure on this normal bundle determined by T_1. The complex structure on $\eta_1 \oplus \dots \oplus \eta_{(p-1)/2}$ together with the orientation of Y determine a compatible orientation on F. With the similar orientation of $q^{-1}(F)$ we see $q^{-1}(F) \longrightarrow F$ has degree $+p$.

Now let $F^{2(k-s)} \subset F$ be a non-empty component of co-dimension s. Then over $F^{2(k-s)}$ we have $\dim(\eta_1) = n_1, \dots, \dim(\eta_{(p-1)/2}) = n_{(p-1)/2}$ with $n_1 + \dots + n_{(p-1)/2} = s$. This defines $\big[\eta_1 \oplus \dots \oplus \eta_{(p-1)/2} \longrightarrow F^{2(k-s)}\big] \in \Omega^{SO}_{2(k-s)}\big(BU(n_1)\times\dots\times BU(n_{(p-1)/2})\big)$. Similarly, there is $\big[q^{-1}(\eta_1) \oplus \dots \oplus q^{-1}(\eta_{(p-1)/2}) \longrightarrow q^{-1}(F^{2(k-s)})\big]$. We claim $p\big[\eta_1 \oplus \dots \oplus \eta_{(p-1)/2} \longrightarrow F^{2(k-s)}\big] = \big[q^{-1}(\eta_1) \oplus \dots \oplus q^{-1}(\eta_{(p-1)/2}) \longrightarrow q(F^{2(k-s)})\big]$ and that the proof is entirely analogous to the proof that $p[Y] = [C]$. That is, since the integral homology of $BU(n_1)\times\dots\times BU(n_{(p-1)/2})$ has no torsion, an element in $\Omega^{SO}_{2(k-s)}\big(BU(n_1)\times\dots\times BU(n_{(p-1)/2})\big)$ is uniquely determined by its generalized Pontrjagin numbers and its generalized Whitney-Steifel numbers, [4; §17]. Now $q^*: H^*(F^{2(k-s)}) \longrightarrow H^*\big(q^{-1}(F^{2(k-s)})\big)$ will carry any characteristic classes of $\big[\eta_1 \oplus \dots \oplus \eta_{(p-1)/2} \longrightarrow F^{2(k-s)}\big]$

into the corresponding expression for $\left[q^{-1}(\eta_1)\oplus\ldots\oplus q^{-1}(\eta_{(p-1)/2}) \longrightarrow q^{-1}(F^{2(k-s)})\right]$ and $q_*: H_{2(k-s)}\left(q^{-1}(F^{2(k-s)}); Z\right) \longrightarrow H_{2(k-s)}(F^{2(k-s)}; Z)$ has degree $+p$, thus the value of any generalized characteristic number of $\left[q^{-1}(\eta_1)\oplus\ldots\oplus q^{-1}(\eta_{(p-1)/2}) \longrightarrow q^{-1}(F^{2(k-s)})\right]$ is equal to p times the corresponding invariant of $\left[\eta_1\oplus\ldots\oplus\eta_{(p-1)/2} \longrightarrow F^{2(k-s)}\right]$. Hence,

$$\begin{aligned} &p\left[\eta_1\oplus\ldots\oplus\eta_{(p-1)/2} \longrightarrow F^{2(k-s)}\right] \\ &\qquad = \left[q^{-1}(\eta_1)\oplus\ldots\oplus q^{-1}(\eta_{(p-1)/2}) \longrightarrow q^{-1}(F^{2(k-s)})\right] \end{aligned}$$

in $\Omega^{SO}_{2(k-s)}\left(BU(n_1)\times\ldots\times BU(n_{(p-1)/2})\right)$. A similar formula is valid for every component of F. Thus, the fixed point data of $\left[T_1, C\right]$ is equal to the fixed point data of $\left[T, Y\right]$ multiplied by p. Since $p[Y] = [C]$ this proves $p\left[T, Y\right] = \left[T_1, C\right] \in \mathcal{O}^{SO}_{2k}(Z_p)$.

Next, Y_f is the quotient of C by the identification $(t, y) \sim (t\lambda^{-j}, T^j(y))$, $0 \leq j < p$, and $q_f: (T_1, C) \longrightarrow (T_f, Y_f)$ is an equivariant cyclic covering. Orient Y_f so that q_f has degree $+p$. Repeat the above argument to show $p\left[T_f, Y_f\right] = \left[T_1, C\right] \in \mathcal{O}^{SO}_{2k}(Z_p)$ also. Since $\mathcal{O}^{SO}_{2k}(Z_p)$ has no odd torsion, it follows

$$\left[T_f, Y_f\right] = \left[T, Y\right] \in \mathcal{O}^{SO}_{2k}(Z_p)$$

and the proof of lemma 12.1 is complete. It might be noted that we have seen examples of derived actions for which Y and Y_f are not of the same homotopy type, yet $\left[T_f, Y_f\right] = \left[T, Y\right]$ anyway.

12.2. Lemma: If (Z_p, Y) is a group of orientation preserving diffeomorphisms on a closed oriented 2k-manifold and if (Z_1, Y_f) is a derived action, then $Tr(Z_p, Y) = Tr(Z_p, Y_f) \in Z(\lambda)$.

Proof: When $T_2(Z_p, Y)$ is defined in terms of the induced representation of Z_p on $H^k(Y; R)$ it is found to be a bordism invariant so we apply Lemma 12.1. For the bordism invariance we refer the reader to [3 ; Th. 5.1].

Now we must assume (Z_p, Y, y_0) is an orientation preserving group of diffeomorphisms on a closed oriented aspherical 2k-manifold with at least one fixed point. With the aid of T_* on $\pi = \pi_1(Y, y_0)$ we introduce the semi-direct product $N = \pi \circ Z_p$.

12.3. Lemma: There is a natural isomorphism $H^1(Y/Z_p; Z) \simeq H^1(N; Z)$.

Proof: There is a spectral sequence $\{E_r^{s,t}, d_r\} \Longrightarrow H^*(N; Z)$ with

$$E_2^{s,t} = H^s(Y^*/N, \underset{\sim}{h}^s)$$

since Y^* is contractible. For $y \in Y^*$ the stalk of the sheaf $h^s \longrightarrow Y^*/N$ at $\nu(y)$ is $H^s(N_y; Z)$, where $N_y \subset N$ is the isotropy subgroup at y. Since N_y is finite $\underset{\sim}{h}^1 \longrightarrow Y^*/N = Y/Z_p$ is the 0-sheaf, and thus

$$H^1(Y/Z_p; Z) = E_2^{1,0} \simeq H^1(N; Z)$$

by the edge homomorphism, see [6; § 9].

As an immediate corollary there is a natural transformation of $H^1(N; Z)/pH^1(N; Z)$ onto the set of strict equivalence classes of derived actions of (Z_p, Y, y_0).

Consider now the exact coefficient sequence in the form

$$0 \longrightarrow H^1(N; Z)/pH^1(N; Z) \longrightarrow H^1(N; Z_p) \xrightarrow{\delta^*} H^2(N; Z) \xrightarrow{p} H^2(N; Z).$$

If $a \in A(p)$, then $pa = 0$ and there is a $b \in H^1(N; Z_p) = \mathrm{Hom}(N, Z_p)$ with $\delta^*(b) = a$. Furthermore for every finite subgroup $K \subset N$, $i_K^*(a) \subset H^2(K; Z) \simeq Z_p$ is a generator. This implies that the kernel $\pi^b \subset N$ of $b: N \longrightarrow Z_p$ is torsionless and hence b is an epimorphism. Note that $Y^*/\pi^b = Y(b)$ is a closed oriented aspherical manifold of dimension $2k$ on which there is induced by $b: N \longrightarrow Z_p$ an action $(Z_p, Y(b))$.

12.4. Lemma: The value of $\mathrm{Tr}(Z_p, Y(b))$ depends only on $\delta^*(b) = a \in H^2(N; Z)$.

Proof: If $\delta^*(b') = a$ also then $b - b'$ is the image of a unique element in $H^1(N; Z)/pH^1(N; Z)$. But then $(Z_p, Y(b'))$ is simply the derived action of $(Z_p, Y(b))$ corresponding to this element. (N.B. $Y/Z_p = Y(b)/Z_p = Y^*/N$, so Lemma 12.3 applies to $(Z_p, Y(b))$ also.) We apply Lemma 12.2.

We let $\mathrm{tr}(a) = \mathrm{Tr}(Z_p, Y(b)) \in Z(\lambda)$ where $\delta^*(b) = a \in A(p)$.

It is possible to define $\mathrm{tr}(a)$ algebraically since $Y(b)$ is aspherical. This is done as follows. Choose any element $\alpha \in N$ with $\alpha^p = 1$ and $b(\alpha) = \lambda \in Z_p$. Such a choice must exist since π^b is torsionless. By conjugation with α we obtain an automorphism group (Z_p, π^b). Obviously π^b is a real oriented Poincaré group of dimension $2k$ since $\pi^b \simeq \pi_1(Y(b))$. Thus

$\mathrm{Tr}(Z_p, \pi^b) = \mathrm{Tr}\big(Z_p, Y(b)\big) \in Z(\lambda)$ is defined. The choice of α here, subject to the stated conditions, is immaterial for choosing such an α uniquely determines a conjugacy class of a subgroup of N which is isomorphic to Z_p and at the same time selects a generator for each representative of this conjugacy class. By our analysis of p-groups acting on aspherical manifolds we find that this conjugacy class corresponds to a unique component of the fixed point set of $\big(Z_p, Y(b)\big)$, [9; Appendix]. Choose a point in this component and let $\big(T_*, \pi_1\big(Y(b), y_b\big)\big)$ be the automorphism induced by the generator $\lambda \in Z_p$. Then $\big(T_*, \pi_1\big(Y(b), y_b\big)\big) \simeq \big(\mu(\alpha), \pi^b\big)$, where $\mu(\alpha)$ is the automorphism induced by conjugating with α. Hence $\mathrm{Tr}\big(\mu(\alpha), \pi^b\big) = \mathrm{Tr}\big(Z_p, Y(b)\big)$.

We now combine the algebraic and geometric viewpoints in

12.5. Theorem: *If* $a \in A(p) \subset H^2(N; Z)$ *and if* $\Phi: N \longrightarrow N$ *is an automorphism, then*

$$\mathrm{tr}\big(\Phi^*(a)\big) = \pm\, \mathrm{tr}(a).$$

Proof: Select any $b \in H^1(N; Z)$ with $\delta^*(b) = a$. Introduce $b' \in H^1(N; Z)$ so that

$$\begin{tikzcd} N \arrow[d, "\Phi"'] \arrow[dr, "b'"] & \\ N \arrow[r, "b"'] & Z_p \end{tikzcd}$$

is a commutative diagram. Clearly $\delta^*(b') = \Phi^*(a)$. Choose $\alpha \in N$, $\alpha^p = e$. $b(\alpha) = \lambda$ and put $\Phi(\alpha') = \alpha$. Then $b'(\alpha') = \lambda$ also. Now $\Phi: \pi^{b'} \simeq \pi^b$ and $\Phi(\alpha'\beta\alpha'^{-1}) = \alpha\Phi(\beta)\alpha^{-1}$ for all $\beta \in \pi^{b'}$, hence we have an equivariant isomorphism $\Phi: (Z_1, \pi^{b'}) \simeq (Z_p, \pi^b)$. We do not know, however, if Φ preserves or reverses the orientations of these two real oriented Poincaré groups. Thus at most we can only assert $\mathrm{tr}\big(\Phi^*(a)\big) = \pm\mathrm{tr}(a)$.

We are principally concerned with comparing a with $m \cdot a$ for some $0 < m < p$. Let us observe $Z(\lambda) \subset Q(\lambda)$ is the ring of algebraic integers in the cyclotomic extension of Q obtained by adjoining the roots of $1+x+\ldots+x^{p-1}$. The Galois group acts on $Z(\lambda)$ therefore and for each q, $0 < q < p$ we denote by $G_q: Z(\lambda) \longrightarrow Z(\lambda)$ the automorphism

$$\sum_{j-1}^{p-1} n_j \lambda^j \longrightarrow \sum_{j=1}^{p-1} n_j \lambda^{qj} .$$

12.6. Corollary: Suppose $a \in A(p)$ and for some m, $0 < m < p$, there is an automorphism $\Phi: N \longrightarrow N$ for which $\Phi^*(a) = ma$, then

$$tr(a) = \pm G_q\big(tr(a)\big)$$

where $qm \equiv 1 \bmod p$.

Proof: Suppose we have selected $b \in H^1(N, Z_p)$ with $\delta^*(b) = a$, then surely $\delta^*(mb) = ma$. Suppose we have also chosen $\alpha \in N$ with $\alpha^p = 1$, $b(\alpha) = \lambda$. Then $(mb)(\alpha) = \lambda^m$ while $(mb)(\alpha^q) = \lambda^{mq} = \lambda$ if $mq \equiv 1 \bmod p$. This replaces $\big(\mu(\alpha), \pi^b\big)$ by $\big(\mu(\alpha)^q, \pi^b\big)$ and $Tr\big(\mu(\alpha), \pi^b\big)$ with $G_q\Big(Tr\big(\mu(\alpha), \pi^b\big)\Big) = tr(ma)$.

Theorem 12.5 is also valid if we begin with a cyclic group of odd prime power order (Z_{p^s}, Y, y_0) acting as a group of orientation preserving diffeomorphisms on a closed aspherical manifold with at least one fixed point. We use $A(p^s) = \{a \,|\, a \in A,\ p^s a = 0\}$. Rather than generalize Lemma 12.1, it is more convenient to give a direct generalization of Lemma 12.2. The value of $Tr(Z_{p^s}, Y)$ depends only on fixed point data, [3, Th. 7.2] thus $Tr(Z_{p^s}, Y_f) = Tr(Z_{p^s}, Y)$ can be proved by showing

$$p^s Tr(Z_{p^s}, Y_f) = Tr(Z_{p^s}, C) = p^s Tr(Z_{p^s}, Y)$$

and this may be done by comparing the data for the set of points under all of Z_{p^s}. The argument is quite analogous to the proof of Lemma 12.1.

We should also recall that $N = \pi \circ Z_{p^s}$ is centerless if T_* has order p^s in $Out(\pi)$ and the only central element of π fixed under T_* is the identity. Recalling our discussion of the groups $L_q \simeq \pi_1\big(X(q)\big)$ we may say

12.7. Corollary: Suppose (Z_{p^s}, Y, y_0) is a cyclic group of orientation preserving diffeomorphisms on a closed aspherical manifold with at least one fixed point and that $N = \pi_1(Y, y_0) \circ Z_{p^s}$ is centerless. If for some integer q, with $(q, p) = 1$, $\pi_1\big(X(q)\big) \simeq \pi_1\big(X(1)\big)$ then $tr(a) = \pm G_q\big(tr(a)\big)$ where $a \in A(p^s)$ is the image of $0 \longrightarrow Z \xrightarrow{p^s} Z \longrightarrow Z_{p^s} \longrightarrow 0$ under $H^2(Z_{p^s}; Z) \longrightarrow H^2(N; Z)$.

If $\pi_1\big(X(q)\big) \simeq \pi_1\big(X(1)\big)$ then since N is centerless there is an automorphism $\Phi: N \longrightarrow N$ with $\Phi^*(a) = \pm ma$ where $mq \equiv 1 \bmod p^s$.

13. Another type of example

We shall construct some pairs of closed oriented manifolds with isomorphic homotopy groups which are not of the same homotopy type. These manifolds can be regarded as skeletal approximations for the classifying spaces of certain Fuchsian groups. For the first six lemmas we have chosen to utilize the techniques of §8 rather than the more ad hoc method of §9. The reader familiar with the techniques of §9 and §10 may easily furnish alternate arguments for these first six lemmas. This latter method will yield slightly more general examples than treated herein.

Fix an odd prime p and let us return to the algebraic curve $Y \subset CP(Z)$ given by

$$Y = \left\{ [z_1, z_2, z_3] \,\middle|\, z_1^p + z_2^p + z_3^p = 0 \right\} \quad ,$$

and the map of period p, (T, Y), given by $T[z_1, z_2, z_3] = [\lambda z_1, z_2, z_3]$, $\lambda = \exp(2\pi i/p)$. There are p distinct fixed points $\left\{ [0, \lambda^j, -1] \right\}_{j=0}^{p-1}$. At each fixed point we receive a local representation of Z_p in the complex tangent line to Y at the point. The local representations are all the same; namely, multiplication by λ. We shall also need to refer to the fact that $Y/Z_p = CP(1)$ is simply connected.

Let $y_0 = [0, 1, -1] \in Y$, put $\pi = \pi_1(Y, y_0)$ and $T_*: \pi \simeq \pi$ the automorphism induced by T. For each integer q, $0 < q < p$, we introduce a semi-direct product $N_q = \pi \circ Z_p$ with multiplication $((\alpha, r))_q((\beta, s))_q = ((\alpha T_*^{rq}(\beta), r+s))_q$. We use $((\ ,\ ))_q$ to indicate that the integers r, s are read modulo p.

13.1. <u>Lemma</u>: <u>There is a canonical isomorphism</u> $N_q \simeq N_1$ <u>given by</u> $((\alpha, r))_q \longrightarrow ((\alpha, qr))_1$.

<u>Proof</u>: This is a trivial corollary of the definition.

Now we shall examine $\mathrm{Aut}(N_q)$. Recall that there is a well defined orientation preserving properly discontinuous action (N_q, R^2) with $R^2/N_q = CP(1)$. If we factor out $\pi \subset N_q$ we see that (N_q, R^2) covers (T^q, Y). Our study of semi-direct products [9; Appendix] shows that for each non-trivial finite subgroup $K \subset N_q$ there is a canonical isomorphism $Z_p \simeq K$ given by representing K as the graph of the unique crossed-homomorphism $\phi: Z_p \longrightarrow (T_*^q, \pi)$. With this we obtain a generator $g_K \in K$ of the form $((\delta, 1))_q$ where $\delta T_*^q(\delta) \ldots T_*^{(p-1)q}(\delta) = e$. Under the natural homomorphism $\nu_q : N_q \longrightarrow Z_p$ we see $\nu_q(g_K) = ((1))$. Now g_K has a <u>unique</u> fixed point, $x_K \in R^2$, which under the covering map $R^2 \longrightarrow Y$ is carried into one of the fixed points of (T^q, Y). Thus the real linear representation of K, induced by the linearization of

g_K at x_0 coincides with that of Z_p obtained by linearizing T^q at the corresponding fixed point in Y. Thus the local representation of K in the tangent space at its unique fixed point is independent of the subgroup K. (Indeed, it amounts to multiplication by λ^q.)

13.2. Lemma: *If $\Phi: N_q \simeq N_q$ is an automorphism, then either $\Phi(g_K) = g_{\Phi(K)}$ for every non-trivial finite subgroup, or $\Phi(g_K^{-1}) = g_{\Phi(K)}$ for every non-trivial finite subgroup.*

Proof: According to the Nielsen uniqueness theorem there is a diffeomorphism $\zeta: R^2 \to R^2$ such that $\zeta(gx) = \Phi(g)\zeta(x)$ for all $g \in N_q$, $x \in R^2$. In particular $\zeta(x_K) = x_{\Phi(K)}$ so we compare $d\zeta$ on the tangent space at x_K to that at $x_{\Phi(K)}$. Suppose ζ is orientation preserving. Then $d\zeta$ yields an orientation preserving linear isomorphism of the representation of Z_p determined by g_K to that determined by $\Phi(g_K)$, which is some multiple of $g_{\Phi(K)}$. But the representation determined by $g_{\Phi(K)}$ is also isomorphic to that determined by g_K. Thus the only possibility is $\Phi(g_K) = g_{\Phi(K)}$. If ζ reverses orientation then $\Phi(g_K^{-1}) = g_{\Phi(K)}$.

By choosing the isomorphisms $K \simeq Z_p$ we obtain generators $b_K \in H^1(K; Z_p) = \mathrm{Hom}(K; Z_p)$. Since $\delta: H^1(K; Z_p) \simeq H^2(K; Z)$ we may use $\delta(b_K) = a_K$ as the generator of $H^2(K; Z_p)$. Now we may describe the monomorphism

$$0 \longrightarrow \mathrm{Tor}\, H^2(N_q; Z) \longrightarrow H^0\big(CP(1); \underset{\sim}{h}^2\big) = (Z_p)^p .$$

(See [8] or [6; 9.2] for the meaning of $\underset{\sim}{h}^2$.) Select $K_1, \ldots, K_p$ a representative from each of the conjugacy classes of the non-trivial finite subgroups of N_q. To $a \in \mathrm{Tor}\, H^2(N_q; Z)$ we assign an ordered p-tuple of integers $(n_1, \ldots, n_p)$ where $0 \leq n_j < p$ is the unique integer with

$$n_j a_{K_j} = i^*_{K_j}(a)$$

This determines the element in $(Z_p)^p$.

We must note that n_j is independent of K_j in its conjugacy class. Consider from Lemma 13.2 that $hg_K h^{-1} = g_{hKh^{-1}}$ so that under the induced isomorphism

$$h^*: H^2(hKh^{-1}; Z) \simeq H^2(K; Z)$$

we have $h^*(a_{hKh^{-1}}) = a_K$. Finally,

$$\begin{tikzcd}
 & H^2(N_q; Z) \arrow[dl, "i^*_{hKh^{-1}}"'] \arrow[dr, "i^*_K"] & \\
H^2(hKh^{-1}; Z) \arrow[rr, "h^*"'] & & H^2(K; Z)
\end{tikzcd}$$

commutes. Thus n_j depends only on the conjugacy class of n_j. Combining this with Lemma 13.2 we have

13.3. Lemma: *If $\Phi: N_q \simeq N_q$ is an automorphism and $a \in \operatorname{Tor} H^2(N_q; Z)$ then the sequence assigned to $\Phi^*(a)$ is either a permutation of $(n_1, \ldots, n_p)$ or of $(p - n_1, \ldots, p - n_p)$.*

This suggests that we say $a \in \operatorname{Tor} H^2(N_q; Z)$ is *diagonal* if and only if $n_1 = \ldots = n_p$.

13.4. Lemma: *If $a \in \operatorname{Tor} H^2(N_q; Z)$ is diagonal then $\Phi^*(a) = \pm a$ for every automorphisms $\Phi: N_q \simeq N_q$.*

Recall that $H^1(N; Z) \simeq H^1(R^2/N; Z) = 0$ so that $\delta: H^1(N; Z_p) \longrightarrow H^2(N; Z)$ is a monomorphism. If $\nu_q: N_q \longrightarrow Z_p$ is the canonical homomorphism then $\delta(\nu_q)$ is a diagonal element with sequence $(1, \ldots, 1)$. Thus $\nu_q \circ \Phi = \pm \nu_q$ for every automorphism Φ. This means $\pi \subset N_q$, the kernel of ν_q, is preserved by every automorphism.

13.5. Lemma: *The subgroup $\pi \subset N_q$ is a characteristic subgroup.*

Now we can write every automorphism of N_q in the form

$$\Phi((\alpha, r))_q = \big(\!\big(c(\alpha)\phi(r), \delta r\big)\!\big)_q$$

where $c: \pi \longrightarrow \pi$ is an automorphism, δ, $0 \leq \delta < p$ is a unique integer and $\phi: Z_p \longrightarrow (T^{q\delta}, \pi)$ is a crossed homomorphism. Now observe that since

$$\Phi\big(\!\big((e, 1))_q((\alpha, 0))_q\big)\!\big) = \Phi\big(\!\big(T^q(\alpha), 1\big)\!\big)_q$$

we have $\phi(1) T^{q\delta}\big(c(\alpha)\big) = \big(cT^q(\alpha)\big)\phi(1)$. Thus in $\operatorname{Out}(\pi)$, T^q is conjugate to $T^{q\delta}$. But if $mq \equiv 1 \bmod p$ we have $T^{mq\delta} = T^\delta$ conjugate to T in $\operatorname{Out}(\pi)$. By use of the Atiyah-Bott invariant, however, we showed this is possible if and only if $\delta \equiv \pm 1 \bmod p$. Thus $\delta = 1, p-1$ are the only possible values of δ.

Now if the canonical isomorphism of Lemma 13.1, $N_q \simeq N_1$ is composed with $\mathrm{Aut}(N_1)$ we obtain $\mathrm{Iso}(N_q, N_1)$. Thus for every isomorphism $\phi: N_q \simeq N_1$ we receive a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi \arrow[r] \arrow[d, "c"] & N_q \arrow[r, "\nu_q"] \arrow[d, "\phi"] & Z_p \arrow[r] \arrow[d, "\pm q"] & 0 \\ 1 \arrow[r] & \pi \arrow[r] & N_1 \arrow[r, "\nu_1"] & Z_p \arrow[r] & 0 \end{tikzcd}$$

We must use this in the following form.

13.6. <u>Lemma</u>: <u>For every isomorphism</u> $\phi: N_q \longrightarrow N_1$ <u>there is a commutative diagram</u>

$$\begin{tikzcd} H^*(N_1; Z_p) \arrow[r, "\phi^*"] & H^*(N_q; Z_p) \\ H^*(Z_p; Z_p) \arrow[r, "\pm q"] \arrow[u, "\nu_1^*"] & H^*(Z_p; Z_p) \arrow[u, "\nu_q^*"] \end{tikzcd} .$$

We explain q^* as follows. There is the canonical class $\alpha \in H^1(Z_p; Z_p)$ and $\delta^*(\alpha) = \beta \in H^2(Z_p, Z_p)$. Then $q^*(\alpha) = q\alpha$ so that $q^*(\beta^k) = q^k \cdot \beta^k$, $q^*(\alpha\beta^k) = q^{k+1} \cdot \alpha\beta^k$.

Now we come to the construction of examples. Let $S^{2k+1} \subset C^{k+1}$ be represented by

$$S^{2k+1} = \left\{ (z_1, \ldots, z_{k+1}) \middle| \sum_{j=1}^{k+1} z_j \cdot \bar{z}_j = 1 \right\} .$$

For each pair (q,k) with $0 < q < p$ and $k > 0$ we introduce $M(q,k)$ as the quotient space of $S^{2k+1} \times Y$ with respect to the equivalence $(z_1, \ldots, z_{k+1}; y) \sim (\lambda^r z_1, \ldots, \lambda^r z_{k+1}; T_y^{rq})$. A point in $M(q,k)$ is written $[z_1, \ldots, z_{k+1}, Y]$. There is a fibration $\nu_q: M(q,k) \longrightarrow L(k) = S^{2k+1}/Z_p$ given by $[z_1, \ldots, z_{k+1}; Y] \longrightarrow [z_1, \ldots, z_{k+1}]$ with fiber Y and structure group Z_p. Since y_0 is a fixed point there is also a section $\chi_q: L(k) \longrightarrow M(q,k)$ given by $[z_1, \ldots, z_{k+1}] \longrightarrow [z_1, \ldots, z_{k+1}; y_0]$. Let $x_q = [0, \ldots, 0, 1; y_0] \in M(q,k)$ be the preferred base point. Then since Y is aspherical

$$\pi_1\big(M(q,k), x_q\big) \simeq N_q$$

$$\pi_i\big(M(q,k), x_q\big) \simeq \pi_i(S^{2k+1}), \qquad i > 1.$$

Thus, up to an abstract isomorphism the homotopy groups of $M(q,k)$ are independent of q.

13.7. <u>Lemma</u>: <u>There is an isomorphism</u>

$$\nu_q^*: H^{2k+1}\big(L(k); Z\big) \simeq H^{2k+1}\big(M(q,k); Z\big) \simeq Z.$$

<u>Proof</u>: There is a spectral sequence $\left\{E_r^{s,t}, \delta^r\right\} \Longrightarrow H^*\big(M(q,k); Z\big)$ with $E_2^{s,t} \simeq H^s\big(L(k); H^t(Y;Z)\big)$. Since Y is 2-dimensional, $E_r^{s,t} = 0$ if $t > 0$. Further, $E_2^{2k,1} \simeq H^{2k}\big(Z_p; H^1(Y;Z)\big) = 0$ since T^* leaves only $0 \in H^1(Y;Z)$ fixed. Since T preserves orientation, $E_2^{2k-1,2} \simeq H^{2k-1}(Z_p; Z) = 0$. Thus we have $\nu_q^*: H^{2k+1}\big(L(k); Z \simeq H^{2k+1}\big(M(q,k); Z\big) \simeq Z$.

We have $M(q,k) \subset M(q,k+1)$, so that we obtain $M(q,\infty) = \cup M(q,k)$ which is just the $K(N_q, 1)$. Thus we have

$$\begin{tikzcd} & M(q,\infty) \arrow[dl, "\nu_q"'] \\ L(\infty) & \\ & M(q,k) \arrow[uu, "i_q"'] \arrow[dl, "\nu_q"'] \\ L(k) \arrow[uu, "j_q"] & \end{tikzcd}$$

Let $\sigma_q \in H^{2k+1}(M(q,k); Z)$ be the image of the generator of $H^{2k+1}(L(k); Z)$ under ν_q^*.
Consider

$$H^{2k+1}(N_q; Z_p) \xrightarrow{i_q^*} H^{2k+1}\big(M(q,k); Z_p\big) \xleftarrow{\ \rho\ } H^{2k+1}\big(M(q,k); Z\big)$$

where ρ is reduction mod p. Then

$$\rho(\sigma_q) = i_q^* \nu_q^* (\alpha\beta^k) \ .$$

We come now to

13.8. <u>Theorem</u>: <u>If there is a map</u> $f: \big(M(q,k), x_q\big) \longrightarrow \big(M(1,k), x_1\big)$ <u>which induces an isomorphism of homotopy and homology</u>, <u>then</u> $q^{k+1} = \pm 1 \bmod p$.

<u>Proof:</u> We may extend f to a map $F: M(q,\infty) \longrightarrow M(1,\infty)$ which induces $F_*: \pi_1\big(M(q,\infty)\big) \simeq \pi_1\big(M(1,\infty)\big)$. Thus, according to Lemma 13.6,

$$F^* \nu_1^* (\alpha\beta^k) = \pm q^{k+1} \nu_q^* (\alpha\beta^k).$$

Since

$$\begin{array}{ccc} M(q,\infty) & \xrightarrow{F} & M(1,\infty) \\ \uparrow{\scriptstyle i_q} & & \uparrow{\scriptstyle i_1} \\ M(q,k) & \longrightarrow & M(1,k) \end{array}$$

commutes,

$$f^* i_1^* \nu_1^*(\alpha\beta^k) = \pm q^{k+1} i_q^* \nu_q^*(\alpha\beta^k) .$$

However, $f^*: H^{2k+1}(M(1,k);Z) \simeq H^{2k+1}(M(q,k);Z)$ so $f^*(\sigma_1) = \pm\sigma_q$. Reducing this mod p we find $f^* i_1^* \nu(\alpha\beta^k) = \pm \nu_q^*(\alpha\beta^k) = q^{k+1}\nu(\alpha\beta^k)$ in $H^{2k+1}(M(q,k); Z_p)$. Therefore $q^{k+1} = \pm 1$ mod p.

13.9. <u>Corollary</u>: <u>If</u> M(q, k) <u>and</u> M(1, k) <u>have the same homotopy type</u>, <u>then</u> $q^{k+1} \equiv \pm 1 \pmod p$.

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GROUP ACTION AND BETTI SHEAF

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1. *Introduction.* Let K be a geometric simplicial complex, and x a vertex of K. The homology group $H_q(K,K-x;Z)$ is called the q-th local Betti group of K at x; in the older literature only the rank, called local Betti number, was used, of course. The group itself is independent of the given decomposition of the space of K, in fact a topological invariant of the pair (K,x). It is the $(q-1)$-st $(q\geq 2)$ homology group of the link of x (which is the boundary of the star of x) in any decomposition. It is $\cong Z$ in dimension $q=n$, and $=0$ in dimensions $q\neq n$ in case K is an n-manifold and x is not a boundary point (zero in all dimensions for a boundary point of a manifold). Thus it can be considered as the measure of deviation of K at x from a manifold. From these local Betti groups we can build a Betti sheaf; the title refers to that sheaf, even though, strictly speaking, we will deal with a dual concept (see Section 3). This sheaf enters decisively in the proof of the Poincaré duality property of manifolds [2], [12].

In the present paper we consider a compact Lie group G acting on a locally compact, separable, metric space X, denote $f\colon X\to X/G$ the identification map (which is, by definition, continuous, open, and closed), and consider the question how the Betti groups at $x\in X$ compare to the Betti groups of X/G at fx, in the simplest case, when x is not critical under the action of G.

We recall some known facts about the critical sets of the action of G in Section 2, known facts about Betti sheaves in Sections 3, 4, and come to the question mentioned above in Section 5. Let us add that the questions concerning the Betti sheaf will be stated in a "dual" form, as was suggested in [3], p. 7. We prove Theorem 1 below in this context. Important results [12] on Betti sheaves are generally encumbered with technical difficulties, and the new approach is described (or rather just indicated here) in the hope that it may be a change of pace. Of course, full justification of this method would require a thorough development of elementary facts of Algebraic Topology along the lines suggested in [4].

For simplicity of exposition we require X to be separable, metric. A further restriction on X, or a different category of spaces, could shift the emphasis, or would require a different technique.

However, in case x is critical for the action of G, the comparison of the Betti sheaves at x and fx would be interesting even in a very restricted (say PL) category. But in this case a geometric study of orthogonal actions should come first. This justifies an independent publication of the present paper.

2. *Critical Sets of the Action.* Everything that follows in this section depends on an important and well known theorem of Gleason ([10], p. 222). We repeat, however, some definitions in order to bring out the concept of critical sets in an appropriate form. We denote G_x the stability group of $x \in X$, and $G(x) \cong G/G_x$ its orbit.

Definition 1. We say that the point $x \in X$ *(or, equivalently, the orbit* $G(x)$*, or the point* $fx \in X/G$*) is non-critical under (the action of)* G*, if there is a neighborhood* V *of* fx *in* X/G*, and a homeomorphism*

$$(1) \qquad \phi: V \times M \to U \quad (M = G/G_x \ ; \ U = f^{-1}V)$$

such that $f\phi(v,m) = v$, *and that*

$$(2) \qquad g\phi(v,m) = \phi(v,gm) \quad (v \in V \ ; \ m \text{ coset of } G_x)$$

holds true for every $g \in G$. *Every other point of* X *(orbit, or point of* X/G*) is called critical.*

The reason to introduce this definition is, of course, that by the theorem of Gleason every action has non-critical points, thus, by definition, a non-empty open set of non-critical points. The open set of non-critical points is everywhere dense in X, for the following reasons. Let X_1 be the set of critical points. Suppose that the interior of X_1, denoted int X_1 is $\neq \phi$. Then $(G, \text{int } X_1)$ has a non-critical point, which is also non-critical for (G,X). This is a contradiction and shows int $X_1 = \phi$. Thus we define critical sets $X_0 = X \supset X_1 \supset \ldots \supset X_\alpha \supset \ldots$ by transfinite induction as follows: X_α is the set of all critical points of the action $(G, X_{\alpha-1})$ if $\alpha \geq 1$ is not a limit number; for a limit number β we set $X_\beta = \cap\{X_\alpha : \alpha < \beta\}$. Then X_β is closed in X_α if $\alpha < \beta$, and $X_\alpha - X_{\alpha+1}$ is open and everywhere dense in X_α; $X_\omega = \phi$ for some ordinal ω. It is a routine matter (see [8]) to prove that $X_\alpha - X_{\alpha+1}$ is a disjoint union of coordinate bundles with structure groups in Steenrod's sense. Notice that the total space of such a bundle need not be connected, although each connected component of $X_\alpha - X_{\alpha+1}$ is a bundle, of course.

The point of view above is slightly different from that of [11], where for a fixed, compact sub-group H of G, the author introduces the set of points $X_{(H)}$ where the stability group is conjugate to H. In order to compare these two points of view, we make only the following remark. Our $X_\alpha - X_{\alpha+1}$ is the disjoint union of $X_{(H)}$'s, each open in X_α, but if we consider, for example, $X-X_1$ as we do in Corollary 3 below, the sub-group H is not fixed yet. Our critical sets introduce thus an ordering and classifying of the sub-spaces $X_{(H)}$, which is needed in our results.

3. *Betti Co-Stack.* Let A be the Alexander cochain complex of X with compact supports ("couverture" in [1], [4]). This is different from the Alexander-Spanier cochain complex where the supports are closed, but not necessarily compact; if X is compact, the two are of course identical. Then $A=\sum A^p$, direct sum for $p=0,1,\ldots,$ where an $a\in A^p$ is an equivalence class of integral valued functions $\phi(x_0,\ldots,x_p)$, $x_i\in X$, with compact support $S(a)=S(\phi)\subset X$ (whose complement is the set of points of X in the neighborhood of which ϕ is zero for any choice of the arguments), two functions ϕ,ψ being equivalent if and only if $S(\psi-\phi)=\emptyset$. The coboundary operator $\delta\colon A^p\to A^{p+1}$ is defined by the standard formula. The cup-product of ϕ,ψ will be denoted $\phi\psi$. Recall also $\delta(\phi\psi)=(\delta\phi)\psi+(-1)^p\phi\delta\psi$ (ϕ of degree p). We write $H(X)=\sum H^p(X)$ for the derived ring $H(A)=\sum H^p(A)$. For a compact X, $H^p(X)$ is the p-th Čech integral cohomology group of X.

We recall that if C is a closed sub-space of X, and $U=X-C$ its open complement, then $H(X)\to H(C)\to H(U)\to H(X)$ is an exact sequence, whose first homomorphism is called restriction, the second coboundary $(H^p(C)\to H^{p+1}(U))$ and the last injection, denoted i_{XU}, if necessary. A cocycle $a\in A^p$ with $S(a)\subset U$ determines a cohomology class in $H(U)$, whose image under i_{XU} is $a+\delta A$, i.e., a cohomology class of X. Furthermore, if $U\subset V$ are open sub-spaces of X, the injection

$$i_{VU}\colon H(U)\to H(V)\quad (U\subset V) \tag{3}$$

has functorial properties: i_{UU} is the identity, $i_{WU}=i_{WV}i_{VU}$. Thus we have a functor from the category of open subspaces of X and their inclusion maps to the category of graded rings and degree preserving homomorphisms. In particular, if C is a subset of X and $\{U\colon U\supset C\}$ the directed system of open neighborhoods of C, we can form $\varprojlim$, inverse limit, of the values of the functor (we follow the terminology and notations of [9], p. 133; the $<$ there is $\supset$ here; the $\sigma,\tau,\ldots$

there are $U, V, \ldots,$ here).

Definition 2. Given a closed sub-space C *of* X *, the open neighborhoods* U *of* C *form a directed system and the rings* $H(U)$ *and injection homomorphisms* i_{VU} *in (3) determine*

$$(4) \qquad \mathcal{B}^p(C) = \varprojlim H^p(U) \quad (\mathcal{B}(C) = \sum \mathcal{B}^p(C))$$

(inverse limit, [9], p. 133). In particular, if x *is a point of* X *,* $\mathcal{B}(x)$ *is defined. If* $C \subset D$ *are closed sets, we have a homomorphism*

$$(5) \qquad \mathcal{B}(C) \to \mathcal{B}(D) \quad (C \subset D \text{ closed in } X)$$

obtained by "thinning out" a system $\{h_U : U \supset C\}$ *, i.e., keeping only* h_U*'s for which* $U \supset D$ *. This covariant functor from the category of closed sub-spaces of* X *and their inclusion maps to the category of graded rings and homomorphisms will be called Betti co-stack* $\mathcal{B}$ *of* X*.*

As an example, we consider a C^o n-manifold X , and a point $x \in X$, not on the boundary of X . Then the open neighborhoods of x have a cofinal sub-system consisting of those neighborhoods which are open balls in a fixed coordinate patch containing x . In degree $p \geq 1$ the compact cohomology of an open n-ball E^n , being isomorphic to the one point compactification S^n of E^n and all the injections being isomorphisms, we have: $\mathcal{B}^p(x)$ is $=0$ if $p \neq n$ and $\cong Z$, if $p=n$.

In terms of cocycles, our definition gives the following. Given a closed sub-space C of X , we form all cocycles a which "can be pushed in an arbitrary neighborhood of C ", i.e., are such that, given an arbitrary open U containing C there is a $b \in A$ such that $S(a+\delta b) \subset U$. Let $\mathcal{C}$ be the ring of these cocycles a . We denote $\mathcal{D}$ the subset of $\mathcal{C}$ for which it is true that, if $S(a+\delta b) \subset U$, then $a+\delta b = \delta g$ with a $g \in A$ such that $S(g) \subset U$. It can be proved that $\mathcal{B}(C)$ is isomorphic to $\mathcal{C}/\mathcal{D}$. We will not prove this presently, but give a better result for $\mathcal{B}(x)$ in Theorem 1 below.

In order to illustrate this interpretation with cocycles, we consider again the Betti co-stack of a C^o n-manifold X . In case $X=R^n$, let be given a point $x=(\xi_1, \ldots, \xi_n)$ and let us denote α_i the characteristic function of the open half-space where the i-th coordinate is $>\xi_i$. Then, considering α_i as integral valued Alexander-Spanier 0-cochain, $\delta\alpha_i$ is defined, and $S(\delta\alpha_i)$ is the hyperplane where the i-th coordinate is $=\xi_i$. We set

$$(6) \qquad a_x = \delta\alpha_1 \ldots \delta\alpha_n \quad (=\delta(\alpha_1\delta\alpha_2 \ldots \delta\alpha_n)$$

which is an Alexander cocycle whose support $S(a_x)$ is the single point x. (As usual, we will write sometimes $S(a) = x$ instead of $S(a) = \{x\}$.) Any two such cocycles a_x, a_y are cohomologous, in particular there is a $b \in A^{n-1}$ whose support is the segment from x to y and for which $\delta b = a_y - a_x$. Furthermore, the class of a_x is a generator of $H(R^n)$ (see [4], p. 123). This construction being "local", it can be carried out in any C° n-manifold X. Thus we get $\mathcal{B}^p(x)$ for any x not on the boundary of X: in a fixed coordinate path containing x we construct the cocycle (6); if h_U denotes the class of (6) considered a cocycle of U, $x \in U$, then $\{h_U\}$ is a generator of $\mathcal{B}(x)$.

This construction is somehow more general than it seems to be, as shown by the following result.

Theorem 1. Let X *be a locally compact, separable, metric space. Given an element* $h \in \mathcal{B}^P(x)$, $p \geqslant 0$, *of the inverse limit* (4) *there is an Alexander cocycle* $a \in A^P$ *of* X *such that its support is the point* x *(or* $= \phi$ *only in case* $h=0$ *), and that* $h = \{h_U\}$, *where* h_U *is the class of* a *in* $H(U)$, U *open neighborhood of* x; a *thus "represents"* h. *Two such cocycles* a, b $(S(a), S(b) \subset \{x\})$ *represent the same* h, *if and only if for every open* U *containing* x, *there is a* $g \in A^{p-1}$, $S(g) \subset U$, *such that* $b - a = \delta g$.

Remarks. $S(a) \subset \{x\}$ means $S(a) = \{x\}$ or $S(a) = \phi$; in the second case $a=0$, by definition. --In view of the first statement, one may wonder whether the second statement could be inproved by considering only g's with $S(g) = x$. This does not seem to be the case.-- For $p=0$ the theorem implies that $\mathcal{B}^\circ(x)$ is $\neq 0$ only in case x is an isolated point ($\{x\}$ an open sub-space of X), in which case it is $\cong Z$.

Proof. Let U_n be the open set of points at a distance $<1/n$ from x, $n=1,2,\ldots$. This system of neighborhoods being cofinal with the system of all neighborhoods of x, h is given by $h_n \in H^p(U_n)$, where $h_n = i_n h_{n+1}$, $i_n : H(U_{n+1}) \to H(U_n)$ being the injection homomorphism. Let us select in h_n a cocycle c_n such that $x \notin S(c_n)$. This is possible because of the Poincaré axiom ([4], p. 109; in Leray's notations, we write $xc_n = 0$). We now "thin out" the sequences c_n, U_n, as follows. Let n_1 be the first integer such that $S(c_1) \cap \bar{U}_{n_1} = \phi$. Similarly, to c_{n_1} we find an n_2 such that $S(c_{n_1}) \cap \bar{U}_{n_2} = \phi$, etc. To simplify notation, we write again $c_1, c_2, \ldots,$ instead of $c_1, c_{n_1}, \ldots$. Thus we have a sequence of cocycles c_n with the following properties: (a) $\{U_n\}$ is cofinal with the set of all neighborhoods

of x ; (b) $S(c_n) \subset U_n$; (c) $S(c_n) \cap \overline{U}_{n+1} = \phi$; (d) if h_n is the class of c_n in $H(U_n)$ then $\{h_n\}$ represents h . Thus, in virtue of (d), we can select a $b_n \in A^{p-1}$, such that

$$c_{n+1} - c_n = \delta b_n \quad (S(b_n) \subset U_n) . \tag{7}$$

Let us denote u_n the characteristic function of U_n , considered as element of A° . Then $S(u_n) = \overline{U}_n$, $S(\delta u_n) \subset \overline{U}_n - U_n$. In view of (7), we have

$$c_{n+1} - c_n = \delta(u_n b_n) \tag{8}$$

(namely, $\delta(u_n b_n) = (\delta u_n) b_n + u_n \delta b_n = u_n \delta b_n = \delta b_n$, as $S(\delta u_n) \cap S(b_n) = \phi$) . Finally, we select Alexander function $\gamma_n(x_0,\ldots,x_p)$ whose class is c_n , Alexander function β_n whose class is b_n ; u_n itself is an element of A° , as there is no identification of functions in the definition of A° . With these notations we define the Alexander function ϕ as follows:

$$\phi(x_0,\ldots,x_{p-1}) = \begin{cases} \sum_{k=1}^{\infty} u_k \beta_k(x_0,\ldots,x_{p-1}) , & \text{if } x_0 \neq x ; \\ 0 , & \text{if } x_0 = x . \end{cases} \tag{9}$$

This definition can be justified as follows. Suppose $x_0 \neq x$, and let m be the smallest integer such that $x_0 \notin \overline{U}_m$. Then $u_k \beta_k(x_0,\ldots,x_p)$ $= u_k(x_0)\beta_k(x_0,\ldots,x_0) = 0$ for $k \geqslant m$, hence all but a finite number of terms of the infinite series are zero.

We define now the Alexander function α as follows:

$$\alpha(x_0,\ldots,x_p) = \gamma_1(x_0,\ldots,x_p) + (\delta\phi)(x_0,\ldots,x_p) . \tag{10}$$

We say that the class a of the function α has the properties stated in the theorem. Let us prove first: if $y \neq x$, y has a neighborhood V such that (10) is zero when $x_i \in V$. We select the first integer m such that $y \notin U_m$, and determine V so that $\overline{V} \cap \overline{U}_m = \phi$, γ_{m+1} is identically zero if all the arguments belong to $\overline{V}$, and that equations (7), (8) concerning the representing functions γ_n, β_n hold true identically for $n=1,\ldots,$ m , and all arguments belonging to $\overline{V}$. This implies then, of course that in the infinite series of (9) terms with $k \geqslant m$ are zero. Then we have, for $x_i \in V$, $i=0,\ldots,p$,

$$\alpha(x_0,\ldots,x_p) = \gamma_1(x_0,\ldots,x_p) + \sum_{k=1}^{m} u_k\delta\beta_k(x_0,\ldots,x_p)$$

$$= \gamma_{m+1}(x_0,\ldots,x_p)$$

$$= 0$$

(Equations (7), (8) show that this is a "telescoping sum.") In view of this, $S(a)=S(\alpha)\subset\{x\}$. Equation (10) shows that a is in the class h_n. This completes the proof of the first statement of the theorem; the second is but a restatement of previously formulated facts.

Alternate Construction in a Special Case. Let Y be a compact sub-space of $R^{n-1}\subset R^n$, $x=(0,\ldots,0,1)\in R^n$, and $X = \{(1-t)y+tx\colon y\in Y,\ 0\leqslant t\leqslant 1\}$ the cone over Y with vertex at x. Then, denoting $\mathcal{B}$ the Betti co-stack of X, $\mathcal{B}^p(x)$ is $\cong H^{p-1}(Y)$, $p\geqslant 1$, and the construction of the proof above can be replaced by the following one. Let $r\colon X-\{x\}\to Y$ be the retraction $r(\xi_1,\ldots,\xi_n) = (\xi_1/(1-\xi_n),\ldots,0)$ (here $\xi_n\neq 1$). Given an Alexander function $\phi(y_0,\ldots,y_{p-1})$ of Y representing a cocycle $(S(\delta\phi)=\phi)$, we set

$$(11)\qquad \psi(x_0,\ldots,x_{p-1}) = \begin{cases}\phi(rx_0,\ldots,rx_{p-1}), & \text{if } x_i\neq x,\ i=0,\ldots,p-1;\\ 0, & \text{if one } x_i \text{ is } x.\end{cases}$$

Then $\alpha=\delta\psi$ is a cocycle of X, $S(\alpha)\subset\{x\}$, represents an element of $\mathcal{B}^p(x)$, and all elements are represented this way $(p\geqslant 1)$. It would be easy to describe the relation of this construction with that of the proof of Theorem 1.

4. *Relation with the Betti Sheaf.* A contravariant functor from the category of open sets and inclusion maps of a space to an abelian category is called pre-sheaf, and forming limits and topology, or adding conditions we get sheaves. There is an alternate way of approaching Algebraic Topology via a theory of stacks (see [1], [4], [5]). We consider the category whose objects are closed subsets of X, and maps are inclusion maps of such sets. Let F be a contravariant functor from this category to the category of (graded rings, or) abelian groups. For example, for every C closed in X, we may take $H(C)$, and for inclusion maps $D\subset C$ the restriction homomorphism $H(C)\to H(D)$. Such a functor F is called a stack, if

(12) $\varinjlim F(\overline{V}) = F(C)$ ($\overline{V}$: closed ngd of C).

The functor $C \to H(C)$ satisfies condition (12) thus is a stack. As a second example, take $F(C)=H_q(X,X-C;Z)$, singular q-th homology group, and induced homomorphisms. This last stack F could be called q-th Betti stack of X because it gives the local Betti groups as described in the Introduction in case X is the space of a geometric simplicial complex K . This stack has, however, some undesirable properties.

Our Definition 2 introduces a co-stack (covariant functor from closed sets and inclusion maps to rings and homomorphisms), from which stacks could be deduced, and which organize a structure related to local groups of the Introduction into stacks.

Let T be a contravariant functor from the category of abelian groups to the same category (for example, Pontrjagin duality). More generally, T could be a contravariant functor whose domain is the set of cohomology rings of ideals of the Alexander cochain complex A and their morphisms with values in an abelian category. In either case $T\mathcal{B}$ can be formed, and it is a stack provided that the continuity condition (12) is satisfied.

As an example, we indicate how the local Betti groups of the Introduction are related to the co-stack $\mathcal{B}$ of Definition 2. Let $X=|K|$ be the space of a simplicial complex, and $x \in |K|$ be a given point. We may suppose that x is a vertex of K , and denote st x the (closed) star of x , and ln x the link of x . It is geometrically evident that x has a fundamental system of open neighborhoods each of which is homeomorphic to st x - ln x and that the inclusion homomorphisms (3) between these cohomology groups are identities. Thus $\mathcal{B}^p(x) \cong H^p(\text{st } x - \text{ln } x)$ holds true. Now $H^q(\text{st } x)=0$, $q \geqslant 1$, thus $H^p(\text{st } x - \text{ln } x) \cong H^{p-1}(\text{ln } x)$. Similarly, $H_q(\text{st } x;Z)=0$, $q \geqslant 1$, hence $H_p(K,K-x;Z) \cong H_{p-1}(\text{ln } x;Z)$. Thus the relation between $H_q(K,K-x)$ and $\mathcal{B}^q(x)$ $(q \geqslant 1)$ is the same as between $H_{q-1}(\text{ln } x;Z)$ and $H^{q-1}(\text{ln } x)$. Now ln x is a compact space, a simplicial complex, hence Čech, singular and simplicial cohomology are isomorphic and are in duality with the simplicial homology.

For stacks the concept of critical sets can be introduced in much the same way as we did in Definition 1 for actions, see [5]. Essential difference is, of course, that a stack need not have non-critical points, in fact there is a stack on the line each point of which is critical. Similarly, we can introduce critical points for a co-stack like $\mathcal{B}$ (see Section 6, below), or else we can consider critical sets

of stacks $T\mathcal{B}$, T contravariant functor as above. We do not intend to follow up presently either of these possibilities systematically, but will prove some results in the next section, which can be then interpreted in the last section as existence theorems of non-critical points in some instances.

5. *Relations between Critical Sets.* The following result will show, in particular, that the "critical behavior" of the co-stack $\mathcal{B}$ over X at $x \in X$, and that over X/G at fx is the same, provided that x is non-critical under the action of G .

Theorem 2. Let G be a compact Lie group acting on the locally compact, separable, metric space X . Let $x \in X$ be a point non-critical under the action of G (see Definition 1) . As above, let X/G be the orbit space $f\colon X \to X/G$ the identification map, $\mathcal{B}_1$, $\mathcal{B}_2$ the Betti co-stacks of X and X/G respectively (see Definition 2). Under these conditions $\mathcal{B}_2^p(fx) \cong \mathcal{B}_1^{p+q}(x)$ where $q = \dim G(x)$.

We will prove this result together with the more technical one as follows:

Theorem 3. We use the hypotheses and notations of Theorem 2 above (in particular, $x \in X$ is non-critical under the action of G) . There is a compact neighborhood P of x such that in the commutative diagram

$$\begin{array}{ccc} \mathcal{B}_1^{p+q}(y) & \longrightarrow & \mathcal{B}_1^{p+q}(P) \\ \uparrow & & \uparrow \\ \mathcal{B}_2^{p}(fy) & \longrightarrow & \mathcal{B}_2^{p}(fP) \end{array} \qquad (y \in P \; ; \; q = \dim G(x)) \; , \tag{13}$$

where the horizontal homomorphisms are those defined in (5) (see Definition 2), the vertical homomorphisms are isomorphisms. The vertical isomorphism on the left is the same as in Theorem 1 for $y = x$.

Proof. This will be a straightforward application of the Künneth theorem in compact cohomology. As x is a non-critical point under the action of G , we may select a homeomorphism ϕ in (1). We set $v_0 = fx$, $m_0 = G_x \in M$. As M is a (real, analytic) manifold, the set of all open neighborhoods of m_o in M has a cofinal sub-set consisting of open balls D centered to m_o in some fixed coordinate patch of M . If W is an arbitrary neighborhood of v_0 in X/G , the family of neighborhoods $\phi(W \times D)$ is cofinal with the set of all neighborhoods of x , and as $H(D) = H^q(D) \cong Z$, the Künneth exact sequence in integral compact cohomology reduces presently to

$$(14) \qquad 0 \to H^p(W)\otimes H^q(D) \to H^{p+q}(W\times D) \to 0$$

the torsion part being zero (tensor is taken over Z). In the last cohomology group we may replace $W\times D$ with the homeomorphic space $\phi(W\times D)$ (recall that $f\phi$ is the projection), thus we get the exact sequence

$$(15) \qquad 0 \to H^p(W)\otimes H^q(D) \to H^{p+q}(\phi(W\times D)) \to 0 .$$

(In the future we will do such substitution of spaces homeomorphic under ϕ without any comment.) Furthermore, from an appropriate proof of the Künneth theorem (for which I am unable to give simple reference) we can deduce the following. Let α be the Alexander function defined in a coordinate patch of $G(x)$ as a_x was defined in (6) with $n=q$ factors. Denote π: $V\times M \to M$ the projection, and set $r=\phi\pi\phi^{-1}:U\to G(x)$. Then r is a retraction. Let $\beta=r^{\#}\alpha$, Alexander-Spanier function of U. If $h\in H^p(W)$ is now given cohomology class, and ϕ an Alexander function representing it, then the cup-product $(f^{\#}\phi)\beta$ represents the image of $h\otimes e$ under the second homomorphism of (15), e being a generator of $H^q(D)$.

If W_0 is an open neighborhood of v_0 contained in W, and D_0 an open ball centered to m_0 and contained in D, then (14) being natural with respect to tensor-products of homeomorphisms (3), we have the following commutative diagram:

$$(16) \qquad \begin{array}{ccccccc} 0 \to & H^p(W_0)\otimes H^q(D_0) & \to & H^{p+q}(\phi(W_0\times D_0)) & \to 0 \\ & \downarrow & & \downarrow & \\ 0 \to & H^p(W)\otimes H^q(D) & \to & H^{p+q}(\phi(W\times D)) & \to 0 \end{array}$$

From the definition (4) of $\mathcal{B}(x)$, and from these diagrams, we deduce then immediately Theorem 2.

We come now to the proof of Theorem 3. Let Q be an arbitrary compact neighborhood of v_0 in $V\subset X/G$ (see (1)). We take a compact ball E centered to m_0 in M, and we define P occurring in Theorem 3 by $P=\phi(Q\times E)$. In order to use the formulas (14), (15), (16) again, we redefine the notations there as follows. Let $W_0\subset W\subset V$ be arbitrary open sets containing Q, and $D_0\subset D$ arbitrary open balls containing E and concentric with it. Then $\mathcal{B}_1(P)$ is the inverse limit

of the groups $H(\phi(V\times D))\cong H(V\times D)$. We have then the exact sequence (15) and the diagram (16) . The statements of Theorem 3 follow then immediately.

6. *Corollaries.* We recall that critical points of stacks are defined as follows ([5], p. 327). A point x is non-critical point of the stack $\mathcal{F}$, if x has closed neighborhoods $W\subset V$ such that the restriction homomorphism $\mathcal{F}(V) \to \mathcal{F}(y)$ is an epimorphism if $y\in V$, and its kernel is independent of y , if $y\in W$. Every other point of X is called critical point of $\mathcal{F}$. Dually, we say that x is a non-critical point of the co-stack $\mathcal{B}$, if x has closed neighborhoods $W\subset V$, such that $\mathcal{B}(y) \to \mathcal{B}(V)$ is a monomorphism for every $y\in V$, and its image is independent of y , if $y\in W$. Every other point of X is called critical point of $\mathcal{B}$.

Corollary 1. We use the hypotheses and notations of Theorem 2, in particular x *is a non-critical point of the action of* G *in* X *. If* x *is also a non-critical point for the Betti co-stack* $\mathcal{B}_1$ *of* X *, then* fx *is a non-critical point of the Betti co-stack* $\mathcal{B}_2$ *of* X/G *, and vice versa, if* $\mathcal{B}_2$ *is not critical at* fx *,* $\mathcal{B}_1$ *is not critical at* x *. If, on the other hand,* x *is critical for one of these co-stacks, it is also critical for the other.*

Let X be a C° n-manifold without boundary. If E is a compact n-ball in X , then $\mathcal{B}^p(E)$ is $=0$ for $p\neq n$, and $\cong Z$ for $p=n$. Furthermore, $\mathcal{B}(y) \to \mathcal{B}(E)$ is an isomorphism for all $y\in E$. Thus the Betti co-stack of X has no critical points. Generalizing these properties, we say: X is a generalized n-manifold (without boundary) in the Betti co-stack sense if (a) the Betti co-stack $\mathcal{B}$ of X has no critical points; (b) $\mathcal{B}^p(y)$ is $=0$ for $p\neq n$ and $\cong Z$ for $p=n$, for every $y\in X$. To compare these spaces with the gm's of [12] would involve us in the handling of several techniques of cohomology theory, and will not be attempted here. We formulate instead some results for this class of spaces.

Corollary 2. If X *is a generalized manifold in the sense above, in particular, if it is a* C° *n-manifold without boundary, then* fx *is not critical for the Betti co-stack of* X/G *, if* x *is not critical for the action of* G *. Thus if* $X_1\subset X$ *is the first critical set of the action of* G *(see third paragraph in Section 2), then* fX_1 *contains the critical set of the Betti co-stack of* X/G *.*

Corollary 3. If X *is a generalized manifold in the sense above, in particular, if it is a* C° *n-manifold without boundary,* X/G *has an open everywhere dense sub-space (containing* $f(X-X_1)$ *) , each*

component of which is a generalized manifold in the sense above.

Corollary 4. If X/G *is a generalized manifold in the sense above, in particular if it is a* C° *n-manifold, each critical point of the Betti co-stack of* X *is also a critical point for the action of* G *in* X.

The following example shows that Corollary 4 is not meaningless. Let $G=S^1$ act on a circular cone X by rigid rotations around the axis of the cone. X/G is then the line R^1 whose Betti co-stack has no critical points. The vertex of the cone is critical both for the action and for the Betti co-stack of the cone.

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ON INFINITE CYCLIC ACTIONS ON CONTRACTIBLE OPEN 3-MANIFOLDS AND STRONG IRREDUCIBILITY

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In this note examples of infinite cyclic actions, each of which is a covering transformation group, are given. Consideration of these examples leads to a concept of strong irreducibility. Some known irreducible contractible open 3-manifolds, including 3-space R^3, are strongly irreducible, while the others are not.

1. Let M be an irreducible contractible open 3-dimensional manifold and G an infinite cyclic transformation group generated by an orientation preserving autohomeomorphism h of M such that for each compact subset C, $h^n(C) \cap C = \emptyset$ for all but a finite number of integers n. Then G is a covering transformation group (Cf. Kinoshita [5]).

The 2-dimensional case was studied by Kerékjártó [3] and Sperner [10]. When $M = R^3$ and $G = R^1$, see Montgomery and Zippin [9].

An example of an autohomeomorphism h of M as above is a standard translation acting on R^3. Actually there are uncountably many mutually inequivalent such kinds of autohomeomorphism of R^3 (see Kinoshita[4], Kinoshita and Sikkema [6], [7]).

The following is another example. Let k be a simple closed curve in S^3, which is locally tame except at one point. Further suppose that k bounds a 2-cell which is locally tame except at one point. For non-trivial examples of such k see Fox and Artin [1]. It is easy to see that the fundamental group of $S^3 - k$ is an infinite cyclic group. Let M be the universal covering space of $S^3 - k$, which is an irreducible contractible open 3-manifold. Let G be the covering transformation group of this covering, which is an infinite cyclic group.

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The following definition is motivated by the example above.

Definition. Let M be an irreducible contractible open 3-manifold and $\overline{M} = M \cup (p)$, the one-point compactification of M. Let S^2 be a 2-sphere and e a point of S^2. Then M is called strongly irreducible, if for each homeomorphism $f: (S^2,e) \to (\overline{M},p)$ that is locally tame at every point of $S^2 - e$, the closure of one of the complementary domains of $f(S^2)$ in $\overline{M}$ is a 3-cell.

Theorem (Harrold and Moise [2]). The 3-space R^3 is strongly irreducible.

On the other hand it is easy to see that the example above, the universal covering space of $S^3 - k$, is not strongly irreducible, unless k is trivial. The Fox-Artin example of irreducible contractible open 3-manifold (see Fox and Artin [1]) is not strongly irreducible, either.

To see that J. H. C. Whitehead's example (Whitehead [11]) of an irreducible contractible open 3-manifold is strongly irreducible, let us begin with the following definition.

Definition. An open 3-manifold M is called a W_1-space, if M is a monotonely increasing union of solid tori T_n $(n = 1,2,\dots)$ such that T_n is contractible in T_{n+1} for every n. We assume that $T_n \subset \mathring{T}_{n+1}$ and T_n is tame in T_{n+1} for every n.

A W-space is defined by McMillan (Cf. [8]), and a W_1-space is a special case of a W-space. A W-space is an irreducible contractible open 3-manifold. J. H. C. Whitehead's example quoted above is a W_1-space.

Theorem. A W_1-space is strongly irreducible.

The proof is given in Section 2.

Problem. Let M be a strongly irreducible contractible open 3-manifold. Let G be an infinite cyclic transformation group generated by an orientation preserving autohomeomorphism h of M such that for each compact subset C of M, $h^n(C) \cap C = \emptyset$ for all but a finite number of integers n. Is M 3-space R^3?

2. Proof of Theorem. Let T_n be a solid torus for every natural number n such that $T_n \subset \mathring{T}_{n+1}$ and T_n is tame and contractible in T_{n+1}. Let $M = \cup_{n=1}^{\infty} T_n$. Further we assume that for each n, T_n is not trivially imbedded in T_{n+1}, i.e., there is no tame 3-cell C such that $T_n \subset C \subset T_{n+1}$. For, if T_n is trivially imbedded in T_{n+1} for infinitely many n, then we have $M = R^3$, which is strongly irreducible.

Now suppose that $f(S^2) \cap \partial T_n$ consists of a finite number of mutually disjoint simple closed curves for every n. Then these simple closed curves on ∂T_n must be trivial or meridians on ∂T_n, since T_m is not trivially imbedded in T_{m+1} for each

Let c be one of these simple closed curves on ∂T_n $(n > 1)$. Then c bounds a 2-cell D on $f(S^2 - e)$. The 2-cell D contains a simple closed curve c_1 of $f(S^2) \cap \partial T_n$ which is innermost. By the usual technique of cutting and pasting we can delete trivial one's on ∂T_n. Then we may assume that c_1 is a meridian and c_1 bounds a 2-cell D_1 in D. The 2-cell D_1 is contained in T_n.

Then, since T_{n-1} is imbedded non-trivially in T_n, $T_{n-1} \cap D_1$ is not empty. Further, since T_{n-1} is contractible in T_n, $\partial T_{n-1} \cap D_1$ consists of at least two simple closed curves. Deleting trivial one's on ∂T_{n-1}, we have at least two meridians c_1' and c_2'' of ∂T_{n-1} in $\partial T_{n-1} \cap D_1$.

Now we prove that there are at least two meridians of ∂T_{n-1} in $\partial T_{n-1} \cap D_1$ such that these two meridians bound two disjoint 2-cells in D_1. Suppose on the contrary that (1) $\partial T_{n-1} \cap D_1$ consists of meridians $c_1', c_2', \dots, c_m'$, (2) $c_1' = \partial D_1', \dots, c_m' = \partial D_m'$, where each of D_i' $(i = 1, \dots, m)$ is a 2-cell on D_1, and (3) $D_1' \subset D_2' \subset \cdots \subset D_m'$. Then, it is easy to see that c_1 bounds a 2-cell D_2 in T_n such that $D_2 \cap T_{n-1}$ consists of at most one meridian on ∂T_{n-1}. Since T_{n-1} is not trivially imbedded in T_n and T_{n-1} is contractible in T_n, we have a contradiction.

We can repeat this process. However, if 2^{n-1} is larger than the number of simple closed curves of $\partial T_1 \cap f(S^2)$, this can not happen. Hence $\partial T_n \cap f(S^2)$ does not contain a meridian of ∂T_n, if n is sufficiently large.

From this it follows that if n is sufficiently large, then each of the simple closed curves of $f(S^2) \cap \partial T_n$ is trivial on ∂T_n. Now let $\overline{M} = M \cup (p)$, the one-point compactification of M. Then $\{\overline{M} - T_n\}$ $(n = 1,2,\ldots)$ is a basis for neighborhoods of p in $\overline{M}$. Since M is irreducible, it follows that the closure of one of the complementary domains of $f(S^2)$ in $\overline{M}$ is a 3-cell (see also Harrold and Moise [2]). Hence the proof is complete.

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